Module 3

3.1. Centroid of area.

Each and every particle forming a body is attracted by the earth towards its centre. These forces of attraction form a system of parallel forces. The resultant of these forces is the weight of the body. The point of application of this resultant force is called centre of gravity of the body. It is the point at which the entire weight of the body can be assumed to be concentrated. A geometric figure does not posses weight and hence the term centre of gravity as applied to geometric area is something of a misnomer. For this reason the term centroid is used instead of centre of gravity for a plane area.

Centroid of an area is the geometric centre of the area, where the entire area can be assumed to be concentrated. This point is denoted by G. The position of G can be located by specifying its co-ordinates with respect to XY reference lines. The distance of centroid from Y axis is denoted by $\bar{x}$ and that from the X axis is denoted by $\bar{y}$. The values $\bar{x}$ and $\bar{y}$ depends on the position of reference axis. Refer Fig. 3.1. The reference axes passing through the centroid are called centroidal axes. The distance of centroid from the centroidal axis is zero. Refer Fig. 3.2.

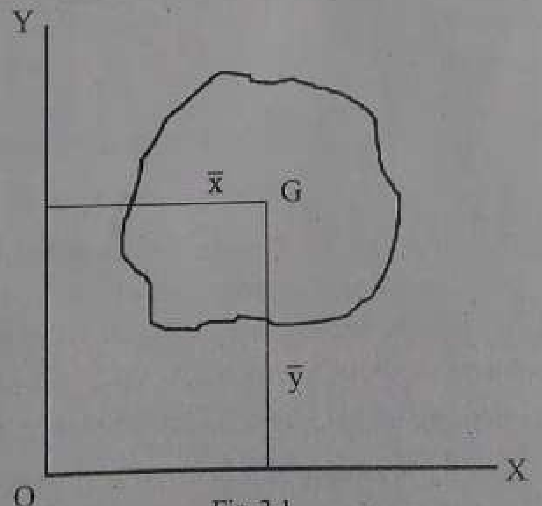

Fig. 3.1

with respect to centroidal axes, $\bar{x} = 0$ and $\bar{y} = 0$. Unless otherwise specified, the horizontal line through the lower most point or line of the area can taken as the X axis and the vertical line passing through the left extreme point or line of the area can be taken as the Y axis. In this case the entire area will be in the first quadrant. Refer Fig. 3.3.

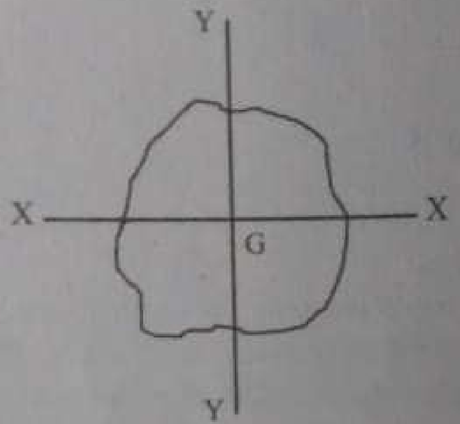

Fig. 3.2

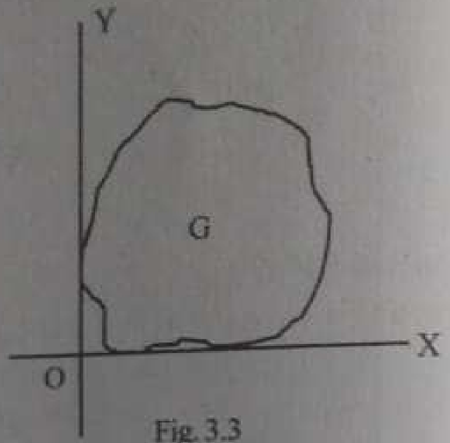

Fig. 3.3

When a plane area has an axis of symmetry, the centroid will be on that axis. When there are two axes of symmetry the point of intersection of the axes of symmetry will be the centroid of the area. Refer Fig. 3.4.

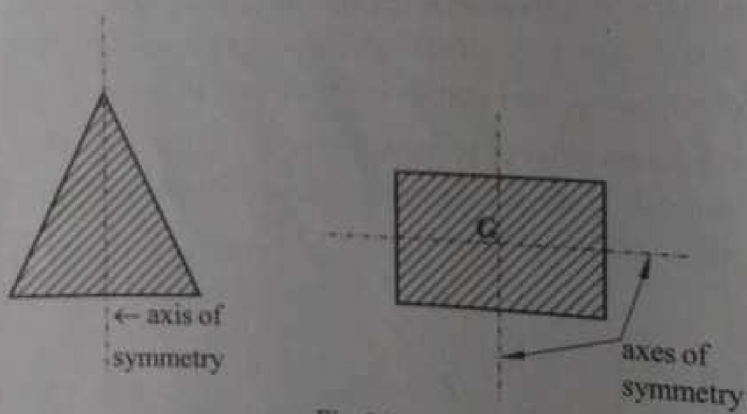

Fig. 3.4

Expressions for $\bar{x}$ and $\bar{y}$.

Consider an area A, as shown in Fig. 3.5. Divide this area into elemental areas a_1, a_2, a_3 etc. Let x_1, x_2, x_3 etc be the distances of elemental areas a_1, a_2, a_3 etc from Y axis and let y_1, y_2, y_3 etc be the distance of elemental areas a_1, a_2, a_3 etc from the X axis. The sum of moments of these elemental areas about Y axis is $a_1x_1 + a_2x_2 + a_3x_3 + ...$ Since the entire area A is assumed to be concentrated at G which is at a distance $\bar{x}$ from the Y axes, the moment of entire area A about the Y axes is $A \times \bar{x} = (a_1 + a_2 + a_3 + ...)\,\bar{x}$. Equating the sum of moments of elemental areas about the Y axis and the moment of entire area about the Y axis,

$$a_1x_1 + a_2x_2 + a_3x_3 + ...=(a_1+a_2+a_3+....)\ \bar{x}$$

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + ...}{a_1 + a_2 + a_3 + ...}$$

$$= \frac{\Sigma\,(ax)}{\Sigma\, a}$$

$$= \frac{\int (ax)}{\int a}$$

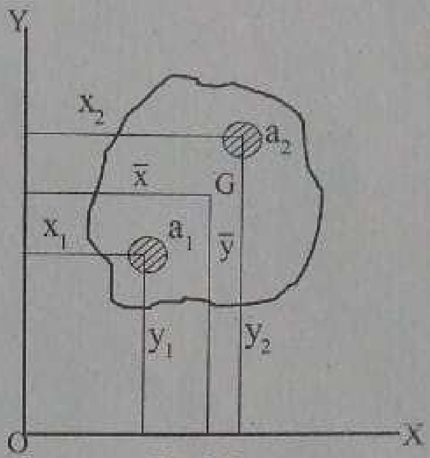

Fig. 3.5

Similarly equating the sum of moments of elemental areas about the Y axis and the moment of entire area about the Y axis,

$$a_1y_1 + a_2y_2 + a_3y_3 + ... = A \times \bar{y}$$

$$= (a_1 + a_2 + a_3 + ...)\ \bar{y}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + ...}{a_1 + a_2 + a_3 + ...}$$

$$= \frac{\Sigma\,(ay)}{\Sigma\, a}$$

$$= \frac{\int (ay)}{\int a}$$

Centroid of rectangle

To find $\bar{y}$

Consider a horizontal strip of thickness dy at a distance y from the X axis as shown in Fig. 3.6.

Elemental area, $dA = b \times dy$

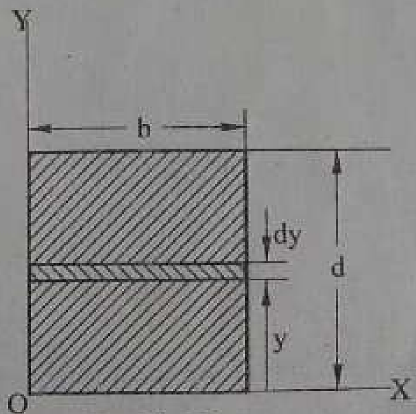

Fig. 3.6

$$\bar{y} = \frac{\int y\, dA}{\int dA}$$

$$= \frac{\int_0^d y\, dA}{\int_0^d dA} = \frac{\int_0^d y\, b\, dy}{\int_0^d b\, dy}$$

$$= \frac{b\left[\frac{y^2}{2}\right]_0^d}{b\,[y]_0^d} = \frac{\frac{b}{2}\, d^2}{b\, d} = \frac{d}{2}$$

$$\bar{y} = \frac{d}{2}$$

To find $\bar{x}$

Consider a vertical strip of thickness dx at a distance x from the Y axis as shown in Fig. 3.7.

Elemental area, $dA = d \times dx$

$$\bar{x} = \frac{\int x\, dA}{\int dA} = \frac{\int_0^b x\, d \times dx}{\int_0^b d \times dx}$$

$$= \frac{d\left[\frac{x^2}{2}\right]_0^b}{d\,[x]_0^b} = \frac{d\,\frac{b^2}{2}}{d\, b} = \frac{b}{2}$$

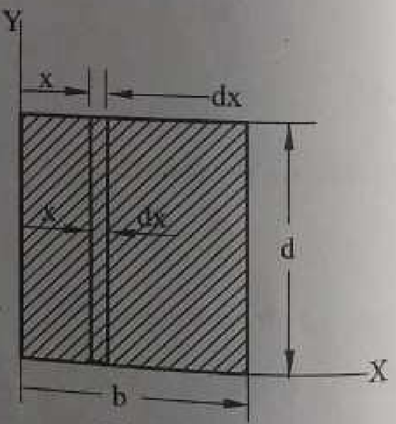

Fig.3.7

$$\bar{x} = \frac{b}{2}$$

When the reference axes, X-X and YY axis, pass through the centre of rectangle,

$\bar{x} = 0$ and $\bar{y} = 0$ as shown in Fig. 3.8

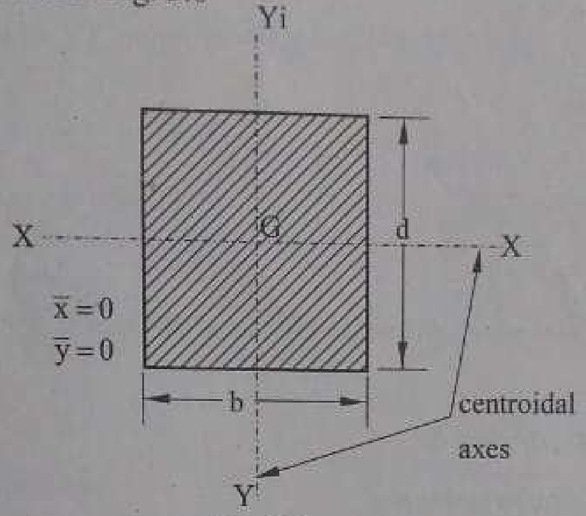

Fig. 3.8

Centroid of a right angled triangle.

Consider right angled triangle of sides b and h. Consider an elemental strip of thickness dy at a distance y from the X - axis. Area of the element, $dA = b_1 \times dy$. Using property of similar triangles, $\frac{b_1}{b} = \frac{(h-y)}{h}$

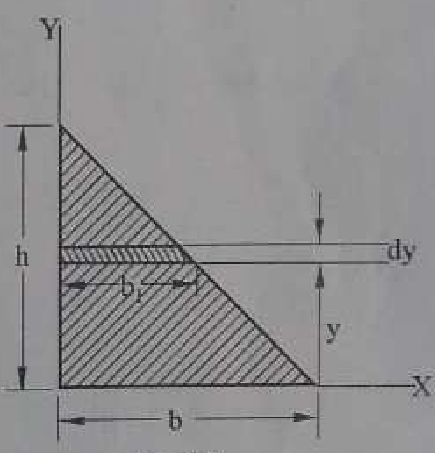

Fig. 3.9

$$b_1 = \frac{b}{h}(h-y)$$

$$\therefore dA = \frac{b}{h}(h-y)\,dy$$

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int_0^h y\frac{b}{h}(h-y)\,dy}{\int_0^h \frac{b}{h}(h-y)\,dy}$$

$$= \frac{\int_0^h (hy - y^2)\,dy}{\int_0^h (h-y)\,dy} = \frac{\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h}{\left[hy - \frac{y^2}{2}\right]_0^h} = \frac{\frac{h^3}{2} - \frac{h^3}{3}}{h^2 - \frac{h^2}{2}}$$

$$= \frac{h^3\left[\frac{1}{2} - \frac{1}{3}\right]}{h^2\left[1 - \frac{1}{2}\right]}$$

$$= \frac{\frac{h^3}{6}}{\frac{h^2}{2}}$$

$$\bar{y} = \frac{h}{3}$$

Consider an elemental strip of thickness dx at a distance x from the Y axis as shown in Fig. 3.10.

Area of the element, $dA = h_1 dx$.

Using property of similar triangles,

$$\frac{h_1}{h} = \frac{(b-x)}{b}$$

$$h_1 = \frac{h}{b}(b-x)$$

$$\therefore dA = \frac{h}{b}(b-x)\,dx$$

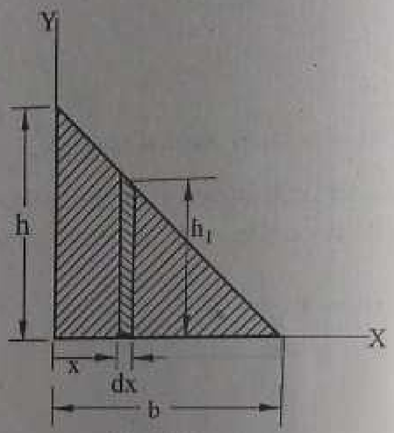

Fig. 3.10

$$\bar{x} = \frac{\int x\,dA}{\int dA}$$

$$= \frac{\int_0^b x\,\frac{h}{b}(b-x)\,dx}{\int_0^b \frac{h}{b}(b-x)\,dx} = \frac{\left[\frac{bx^2}{2} - \frac{x^3}{3}\right]_0^b}{\left[bx - \frac{x^2}{2}\right]_0^b}$$

$$= \frac{\left[\frac{b^3}{2} - \frac{b^3}{3}\right]}{\left[b^2 - \frac{b^2}{2}\right]} = \frac{\frac{b^3}{6}}{\frac{b^2}{2}}$$

$$\bar{x} = \frac{b}{3}$$

Thus, the distance of centroid of a triangular lamina from its base is one third of its altitude.

Example 3.1.

Locate the centroid of the triangular lamina as shown in Fig. 3.11.

Solution

$$\bar{x} = b - \frac{1}{3}b = \frac{2}{3}b$$

$$\bar{y} = h - \frac{1}{3}h = \frac{2}{3}h$$

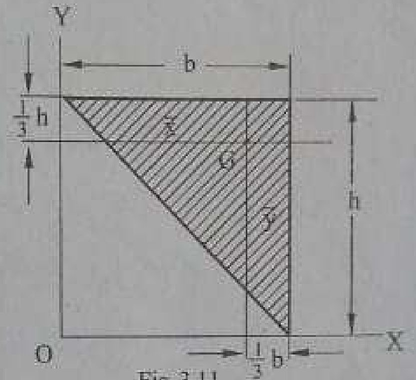

Fig. 3.11

Centroid of isosceles triangle

$$\bar{x} = \frac{b}{2}$$

$$\bar{y} = \frac{1}{3}h$$

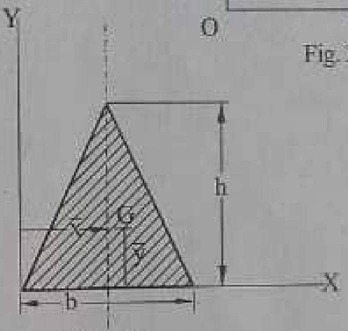

Fig. 3.12

Centroid of a circle

When the reference axes are selected as shown in Fig.3.12,

$$\bar{x} = \bar{y} = \frac{d}{2}$$

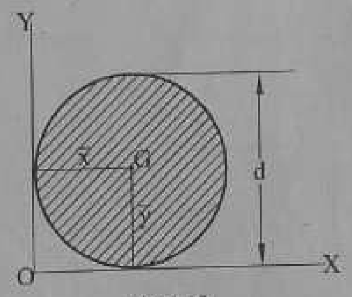

Fig. 3.13

Example 2.23.

Locate the centroid of one quarter of circular area of radius R .

Consider an element as shown in Fig.3.14.

$$dA = r\,d\theta \times dr$$

$$x = r\cos\theta \text{ and}$$

$$y = r\sin\theta$$

$$\bar{x} = \frac{\int x\, dA}{\int dA}$$

$$= \frac{\int_0^{\frac{\pi}{2}}\int_0^R r\cos\theta\, r\, d\theta\, dr}{\int_0^{\frac{\pi}{2}}\int_0^R r\, dr\, d\theta}$$

$$= \frac{\int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^R \cos\theta\, d\theta}{\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^R d\theta}$$

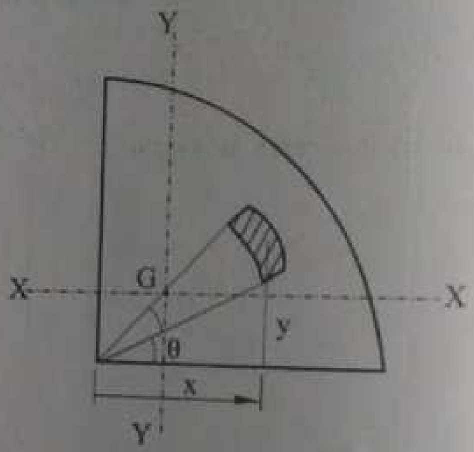

Fig.3.14

$$= \frac{\frac{R^3}{3}\left[\sin\theta\right]_0^{\frac{\pi}{2}}}{\frac{R^2}{2}\left[\theta\right]_0^{\frac{\pi}{2}}}$$

$$= \frac{2R}{3}\frac{\left[\sin\frac{\pi}{2} - \sin 0\right]}{\left[\frac{\pi}{2} - 0\right]} = \frac{4R}{3\pi}$$

$$\bar{y} = \frac{\int y\, dA}{\int dA} = \frac{\int_0^{\frac{\pi}{2}}\int_0^R r\sin\theta\, r\, d\theta\, dr}{\int_0^{\frac{\pi}{2}}\int_0^R r\, d\theta\, dr}$$

$$= \frac{\int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^R \sin\theta \, d\theta}{\int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_0^R d\theta}$$

$$= \frac{\frac{R^3}{3}\left[-\cos\theta\right]_0^{\frac{\pi}{2}}}{\frac{R^2}{2}\left[\theta\right]_0^{\frac{\pi}{2}}}$$

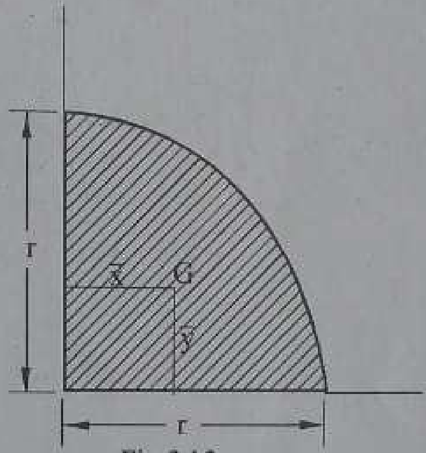

Fig. 3.15

$$= \frac{-\frac{2R}{3}\left[0-1\right]}{\frac{\pi}{2}}$$

$$\bar{y} = \frac{4R}{3\pi}$$

Example 3.3.

Locate the centroid of one quadrant of a circle of radius r as shown in fig. 3.16.

Solution

$$\bar{x} = r - \frac{4r}{3\pi}$$

$$\bar{y} = r - \frac{4r}{3\pi}$$

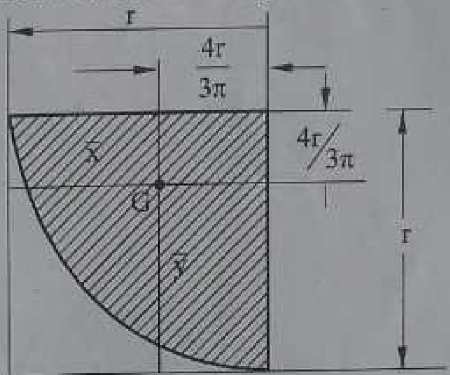

Fig. 3.16

Example 3.4.

Locate the centroid of sector of a circle of radius R with an included angle of α.

Consider an elemental area as shown in Fig. 3.17.

Area of the element, $dA = r\,d\theta \times dr$

$$x = r\cos\theta$$

Since the area is symmetrical with respect to X axis, $\bar{y} = 0$

$$\bar{x} = \frac{\int x\, dA}{\int dA}$$

$$= \frac{\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \int_0^R r\cos\theta\, r\, d\theta\, dr}{\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \int_0^R r\, d\theta\, dr}$$

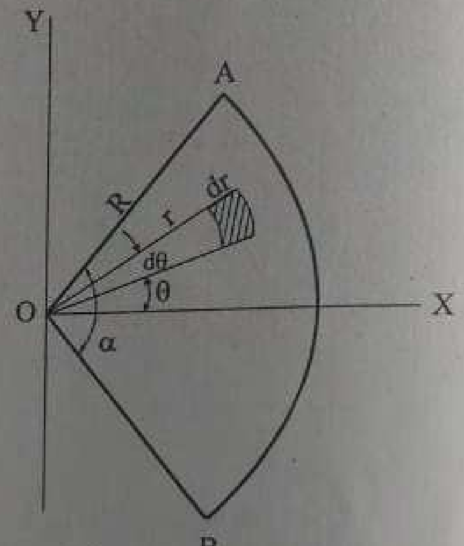

Fig. 3.17

$$= \frac{\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \left[\frac{r^3}{3}\right]_0^R \cos\theta\, d\theta}{\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \left[\frac{r^2}{2}\right]_0^R d\theta}$$

$$= \frac{\frac{R^3}{3}\left[\sin\theta\right]_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}}}{\frac{R^2}{2}\left[\theta\right]_{\frac{-\alpha}{2}}^{\frac{\alpha}{2}}}$$

$$= \frac{2R}{3}\frac{\left[\sin\frac{\alpha}{2} - (-\sin\frac{\alpha}{2})\right]}{\frac{\alpha}{2} - (-\frac{\alpha}{2})}$$

$$= \frac{4R\sin\frac{\alpha}{2}}{3\alpha}$$

Example 3.5.

Locate the centroid of semicircular area of radius R.

Since the area is symmetrical with respect to Y axis, $\bar{x} = 0$

Consider an elemental area as shown in Fig. 3.18

Area of the element $dA = r\,d\theta\,dr$

and $y = r \sin\theta$

$$\bar{y} = \frac{\int y\,dA}{\int dA}$$

$$= \frac{\int_0^{\pi}\int_0^{R} r\sin\theta \times r\,d\theta\,dr}{\int_0^{\pi}\int_0^{R} r\,dr\,d\theta}$$

$$= \frac{\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^R \sin\theta\,d\theta}{\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^R d\theta}$$

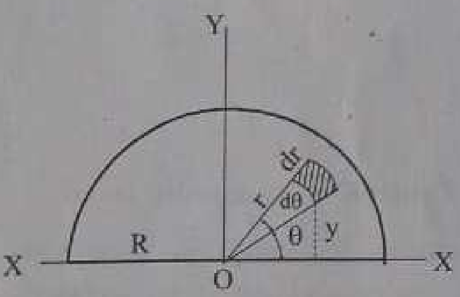

Fig. 3.18

$$= \frac{\frac{R^3}{3}\left[-\cos\theta\right]_0^{\pi}}{\frac{R^2}{2}\left[\theta\right]_0^{\pi}}$$

$$= -\frac{2R}{3}\frac{[\cos\pi - \cos 0]}{[\pi - 0]} = \frac{4R}{3\pi}$$

$$\bar{y} = \frac{4R}{3\pi}$$

Thus the distance of centroid of a semicircular lamina from its base is $\frac{4R}{3\pi}$

Example 3.6.

Locate the centroid of a semicircular lamina of radius r as shown in Fig. 3.19

Solution..

$$\bar{x} = r - \frac{4r}{3\pi}$$

$$\bar{y} = r$$

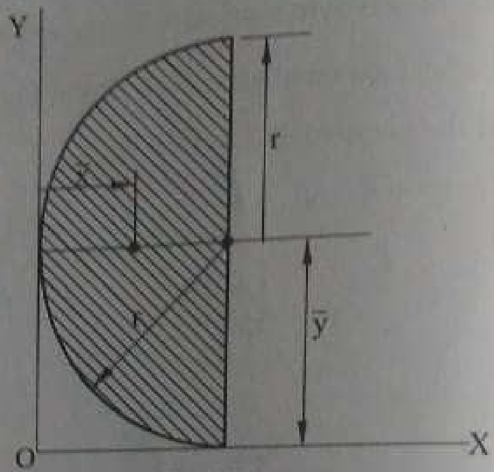

Fig. 3.19

Example 3.7.

Locate the centroid of semicircular lamina of radius r as shown in Fig. 3.20

Solution

$$\bar{x} = r$$

$$\bar{y} = r - \frac{4r}{3\pi}$$

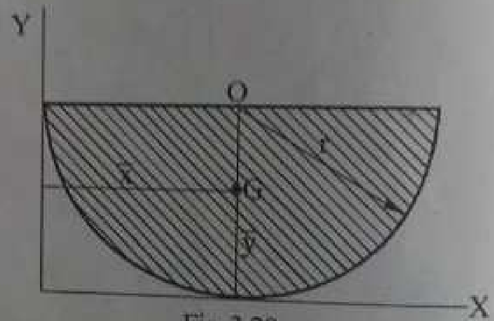

Fig. 3.20

3.2. Centroid of composite areas.

In many engineering applications some composite areas are considered. These composite areas may consists of triangles, rectangles and other simple figures. Hence it is necessary to find the centroid of such composite areas. Composite area can be divided into a number of area like rectangle, triangle, semicircle for which areas and location of centroids of each area are known. The distance of centroid of composite area from Y axis is given by

$$\bar{x} = \frac{\int x\,a}{\int a} = \frac{\Sigma(x\,a)}{\Sigma a} = \frac{a_1x_1 + a_2x_2 + ---}{a_1 + a_2 + ---},$$ where a_1, a_2 etc are area of each section and x_1, x_2 etc are the distance of centroid of a_1, a_2 etc from the Y axis. Similarly,

$$\bar{y} = \frac{\int y\,a}{\int a} = \frac{\Sigma(y\,a)}{\Sigma a} = \frac{a_1y_1 + a_2y_2 + ---}{a_1 + a_2 + ---}$$

Example 3.8 [KTU July 2016]

Locate the centroid of the 'T' section shown in Fig. 3.21.

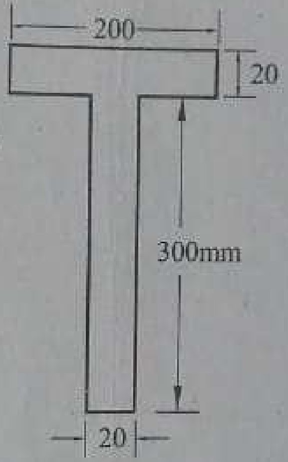

Fig. 3.21

Solution

Since the section is symmetrical with respect to the Y axis, $\bar{x} = 0$.

$$\bar{y} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2}$$

$$a_1 = 300 \times 20 = 6000 \text{ mm}^2$$

$$a_2 = 200 \times 20 = 4000 \text{ mm}^2$$

$$y_1 = \frac{300}{2} = 150 \text{ mm}$$

$$y_2 = 300 + \frac{20}{2} = 310 \text{ mm}$$

$$\therefore \bar{y} = \frac{6000 \times 150 + 4000 \times 310}{6000 + 4000}$$

$$= 214 \text{ mm}$$

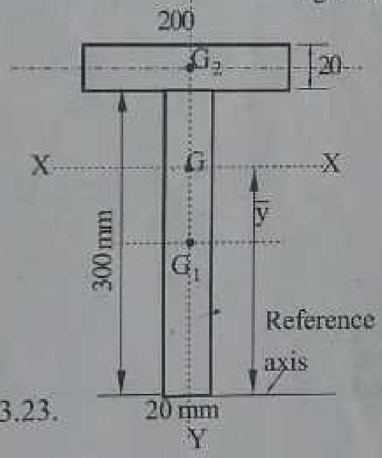

Fig. 3.22

Example 3.9.

Locate the centroid of the area shown in Fig. 3.23.

Solution

$$a_1 = 14 \times 2 = 28 \text{ cm}^2;$$

$$a_2 = 20 \times 2 = 40 \text{ cm}^2$$

$$a_3 = 8 \times 2 = 16 \text{ cm}^2$$

$$x_1 = \frac{14}{2} = 7 \text{ cm}$$

$$x_2 = \frac{2}{2} = 1 \text{ cm}$$

$$x_3 = \frac{8}{2} = 4 \text{cm}$$

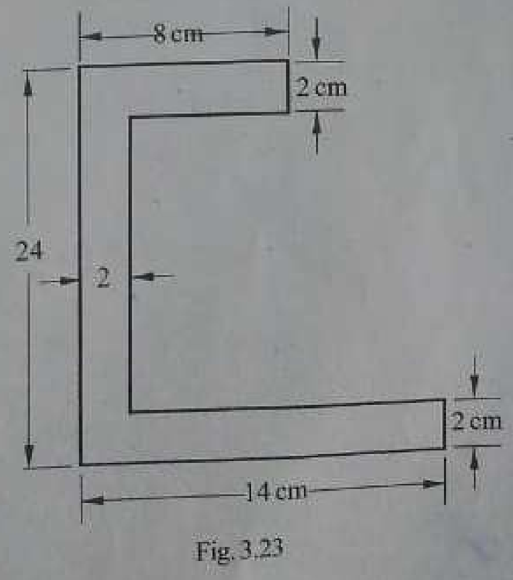

Fig. 3.23

$$y_1 = \frac{2}{2} = 1\text{cm} \quad y_2 = 2 + \frac{20}{2} = 12\text{ cm}$$

$$y_3 = 2 + 20 + \frac{2}{2} = 23\text{cm}$$

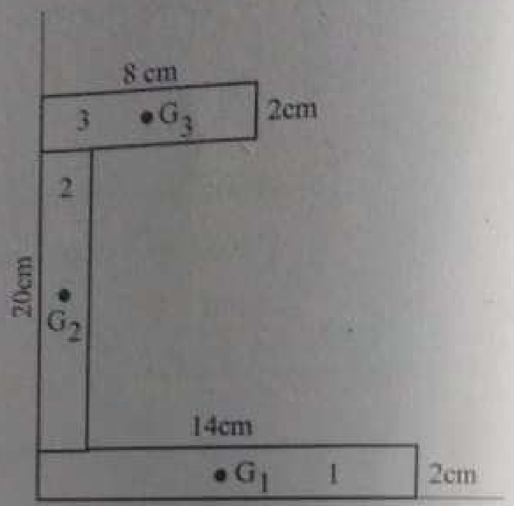

Fig. 3.24.

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

$$= \frac{28 \times 7 + 40 \times 1 + 16 \times 4}{28 + 40 + 16}$$

$$= 3.57 \text{ cm}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3}$$

$$= \frac{28 \times 1 + 40 \times 12 + 16 \times 23}{28 + 40 + 16}$$

$$= 10.43 \text{ cm}$$

Example 3.10.

Determine the centroid of the area shown in Fig. 3.25.

Solution.

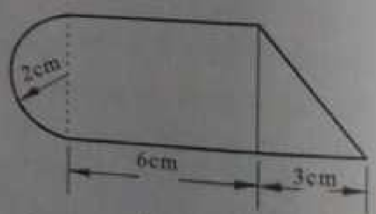

Fig. 3.25

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3}$$

$$a_1 = \frac{\pi r^2}{2} = \frac{\pi}{2} \times 2^2 = 6.28 \text{ cm}^2$$

$$a_2 = 6 \times 4 = 24 \text{ cm}^2$$

$a_3 = \frac{1}{2} \times 3 \times 4 = 6\ \text{cm}^2$

$x_1 = r - \frac{4r}{3\pi} = 2 - \frac{4 \times 2}{3\pi} = 1.15$

$x_2 = 2 + \frac{6}{2} = 5\ \text{cm}$

$x_3 = 2 + 6 + \frac{1}{3} \times 3 = 9\ \text{cm}$

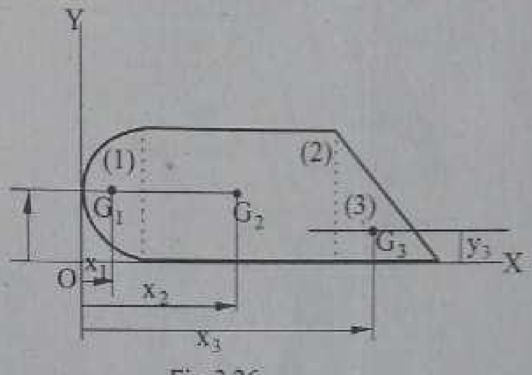

Fig. 3.26

$y_1 = 2\ \text{cm},\ \ y_2 = 2\ \text{cm},\ \ y_3 = \frac{1}{3} \times 4 = 1.33\ \text{cm}$

$$\bar{x} = \frac{6.28 \times 1.15 + 24 \times 5 + 6 \times 9}{6.28 + 24 + 6}$$

$= 5\ \text{cm}$

$$\bar{y} = \frac{6.28 \times 2 + 24 \times 2 + 6 \times 1.33}{6.28 + 24 + 6}$$

$= 1.89\ \text{cm}$

Example 3.11[KTU Jan 2016]

Determine the centre of gravity of the thin homogeneous plane shown in Fig. 3.27

Solution.

$a_1 = \frac{1}{2} \times 100 \times 150$

$= 7500\ \text{mm}^2$

$a_2 = 200 \times 150 = 30000\ \text{mm}^2$

$a_3 = \frac{\pi r^2}{2} = \frac{\pi}{2} \times 50^2 = 3927\ \text{mm}^2$

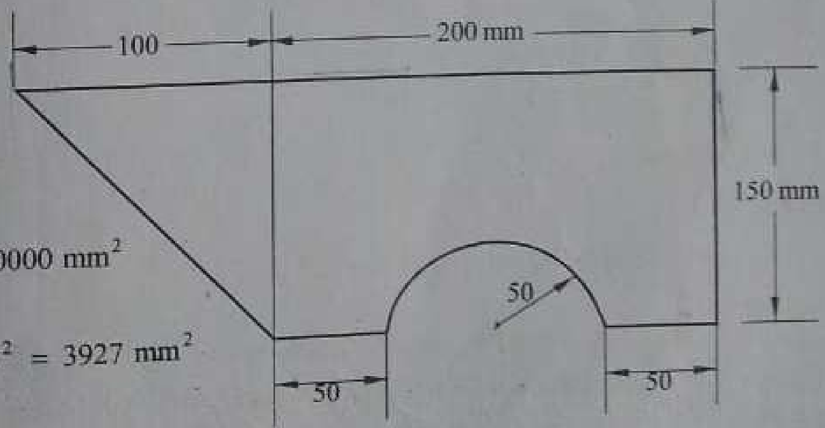

Fig.3.27

$x_1 = (100 - \frac{1}{3} \times 100)$

$= 66.67$ mm

$x_2 = 100 + \frac{200}{2}$

$= 200$ mm

$x_3 = 100 + 50 + 50$

$= 200$ mm

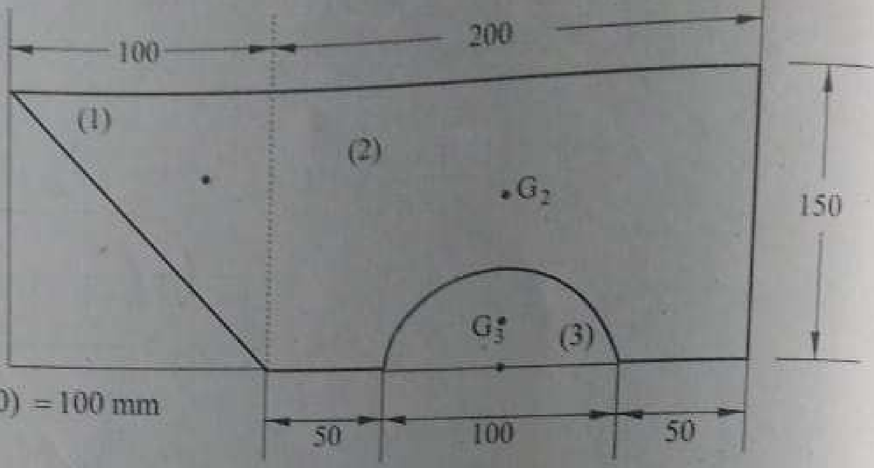

Fig. 3.28

$y_1 = (150 - \frac{1}{3} \times 150) = 100$ mm

$y_2 = \frac{150}{2} = 75$ mm

$y_3 = \frac{4r}{3\pi} = \frac{4 \times 50}{3 \times \pi} = 21.22$ mm

$$\bar{x} = \frac{a_1x_1 + a_2x_2 - a_3x_3}{a_1 + a_2 - a_3}$$

$$= \frac{7500 \times 66.67 + 30{,}000 \times 200 - 3927 \times 200}{7500 + 30000 - 3927}$$

$= 170.21$ mm.

$$\bar{y} = \frac{a_1y_1 + a_2y_2 - a_3y_3}{a_1 + a_2 - a_3}$$

$$= \frac{7500 \times 100 + 30{,}000 \times 75 - 3927 \times 21.22}{7500 + 30{,}000 - 3927}$$

$= 86.88$ mm.

Example 3.12

Locate the centroid of the area shown in Fig. 3.29

Solution.

$$a_1 = \frac{\pi r^2 \times 60}{360} = \pi \times \frac{100^2 \times 60}{360} = 5235.99 \text{ mm}^2$$

$$a_2 = \frac{1}{2} \times 100\cos 60 \times 100\sin 60 = 2165.06\ mm^2$$

$$OG_1 = \frac{4r\sin\frac{\alpha}{2}}{3\alpha} = \frac{4 \times 100 \times \sin\frac{60}{2}}{3 \times \frac{\pi}{180} \times 60} = 63.66\ mm.$$

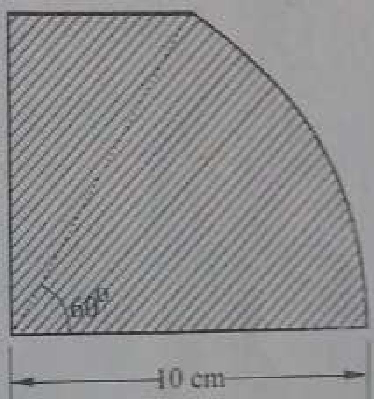

Fig. 3.29

$$x_1 = OG_1 \cos\frac{\alpha}{2}$$

$$= 63.66 \times \cos 30 = 55.13\ mm$$

$$y_1 = OG_1 \sin\frac{\alpha}{2}$$

$$= 63.66 \times \sin 30 = 31.83\ mm.$$

$$x_2 = \frac{1}{3} \times 100\cos 60 = 16.67\ mm$$

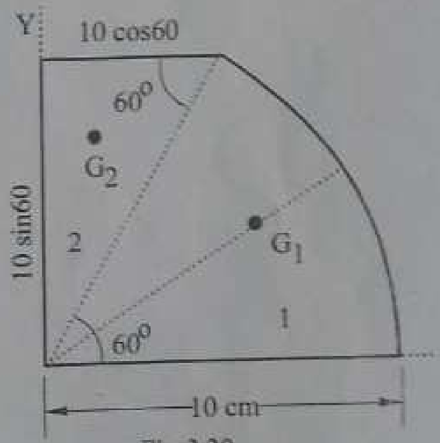

Fig. 3.30

$$y_2 = (100 \times \sin 60 - \frac{1}{3} \times 100\sin 60)$$

$$= \frac{2}{3} \times 100\sin 60 = 57.74\ mm$$

$$\bar{x} = \frac{55.13 \times 5235.99 + 16.67 \times 2165.06}{5235.99 + 2165.06}$$

$$= 43.88\ mm.$$

$$\bar{y} = \frac{31.83 \times 5235.99 + 57.74 \times 2165.06}{5235.99 + 2165.06}$$

$$= 39.41\ mm$$

Example 3.13.

Determine the coordinates of the centre of a 40 mm diameter circle to be cut in a thin plate so that this point will be the centroid of the remaining shaded area shown in Fig. 3.31

Solution.

$$a_1 = 40 \times 60 = 2400\ mm^2$$

$$a_2 = \frac{1}{2} \times 40 \times 60$$

$$= 1200 \text{ mm}^2$$

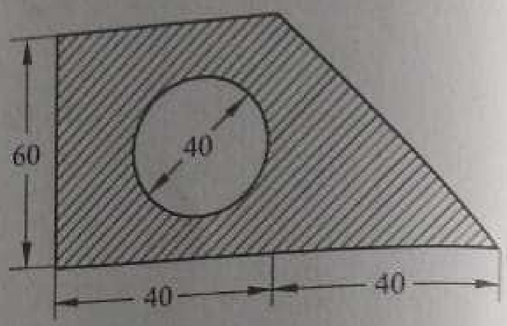

Fig. 3.31

$$a_3 = \pi r^2 = \pi \times 20^2 =$$

$$= 1256.64 \text{ mm}^2$$

$$x_1 = \frac{40}{2} = 20 \text{ mm}$$

$$x_2 = 40 + \frac{1}{3} \times 40 = 53.33 \text{ mm}$$

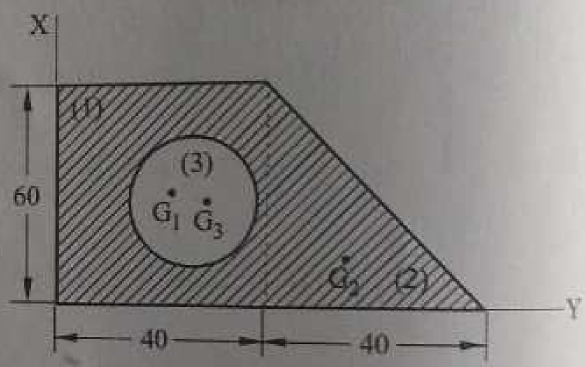

Fig. 3.32

$$x_3 = \bar{x}$$

$$y_1 = \frac{60}{2} = 30 \text{ mm}$$

$$y_2 = \frac{1}{3} \times 60 = 20 \text{ mm}$$

$$y_3 = \bar{y}$$

$$\bar{x} = \frac{a_1x_1 + a_2x_2 - a_3x_3}{a_1 + a_2 + a_3}$$

$$= \frac{2400 \times 20 + 1200 \times 53.33 - 1256.64 \times \bar{x}}{2400 + 1200 - 1256.64}$$

$$2343.36\,\bar{x} + 1256.64\,\bar{x} = 20 \times 2400 + 53.33 \times 1200$$

$$\bar{x} = 31.11 \text{ mm}$$

$$x_3 = \bar{x} = 31.11 \text{ mm}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 - a_3y_3}{a_1 + a_2 - a_3}$$

$$= \frac{2400 \times 30 + 1200 \times 20 - 1256.64 \times \bar{y}}{2400 + 1200 - 1256.64}$$

$\bar{y} = 26.67$ mm

$y_3 = \bar{y} = 26.67$ mm

Example 3.14.

Determine the centroid of the shaded area obtained by cutting a semicircular of diameter 10 cm from the quadrant of a circle of radius 10 cm as shown in Fig. 3.33.

Solution

$$a_1 = \frac{\pi R^2}{4} = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$$

$$a_2 = \frac{\pi r^2}{2} = \frac{\pi \times 5^2}{2} = 39.27 \text{ cm}^2$$

$$x_1 = \frac{4R}{3\pi} = \frac{4 \times 10}{3\pi} = 4.24 \text{ cm}$$

$$x_2 = 5 \text{ cm}$$

$$y_1 = \frac{4R}{3\pi} = 4.24 \text{ cm}$$

$$y_2 = \frac{4r}{3\pi} = \frac{4 \times 5}{3\pi} = 2.12 \text{ cm}$$

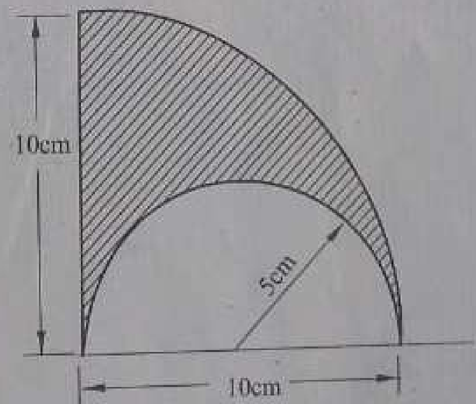

Fig. 3.33

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2}$$

$$= \frac{78.54 \times 4.24 - 39.27 \times 5}{78.57 - 39.27}$$

$$= 3.48 \text{ cm}$$

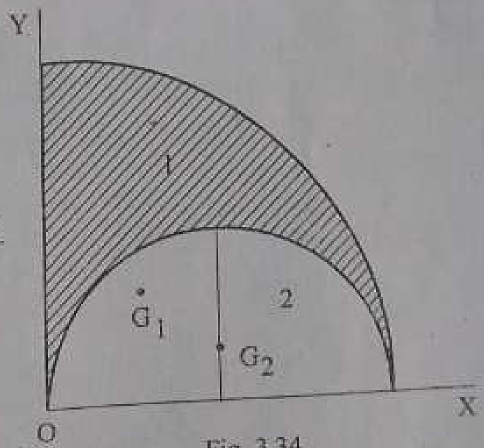

Fig. 3.34

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2}$$

$$= \frac{78.54 \times 4.24 - 39.27 \times 2.12}{78.54 - 39.27}$$

$$= 6.36 \text{ cm}$$

Example 3.15

Locate the centroid of the shaded area as shown in Fig. 3.35

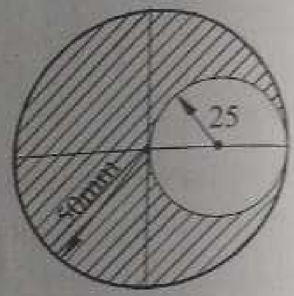

Fig. 3.35

Solution.

$a_1 = \pi R^2 = \pi \times 50^2 = 2500\,\pi\ mm^2$

$a_2 = \pi r^2 = \pi \times 25^2 = 625\,\pi\ mm^2$

$x_1 = R = 50\ mm$

$x_2 = R + r = 50 + 25 = 75\ mm$

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2}$$

$$= \frac{2500\,\pi \times 50 - 625\,\pi \times 75}{2500\,\pi - 625\,\pi}$$

$$= 41.67\ mm$$

$\bar{y} = 0$

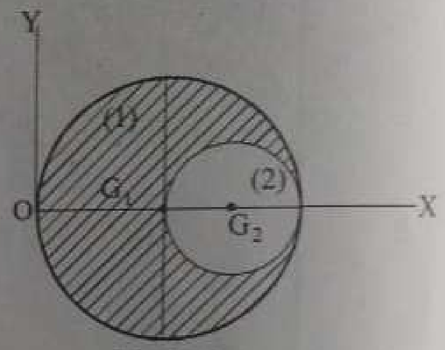

Fig.3.36

Example 3.16

Locate the centroid of the trapezium with parallel side a and b and height h as shown in Fig. 3.37

Solution

$a_1 = a\,h$

$$a_2 = a_3 = \frac{1}{2}\left(\frac{b-a}{2}\right)h$$

$$a_1 + a_2 + a_3 = \left(\frac{a+b}{2}\right)h$$

$$y_1 = \frac{h}{2}$$

$$y_2 = y_3 = \frac{1}{3}h$$

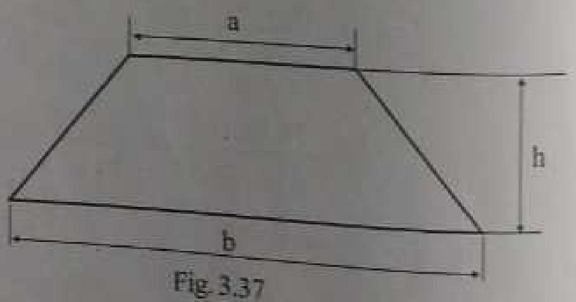

Fig. 3.37

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = \frac{ah \times \frac{h}{2} + 2\left[\frac{1}{2}\left(\frac{b-a}{2}\right)h\right] \times \frac{1}{3}h}{\left(\frac{a+b}{2}\right)h}$$

$$= \frac{\frac{a\,h^2}{2} + (b-a) \times \frac{h^2}{6}}{\left(\frac{a+b}{2}\right)h}$$

$$= \frac{ah + (b-a)\frac{h}{3}}{(a+b)}$$

$$= \frac{3ah + bh - ah}{3(a+b)} = \frac{2ah + bh}{3(a+b)}$$

$$\bar{y} = \frac{(2a+b)}{(a+b)} \times \frac{h}{3}$$

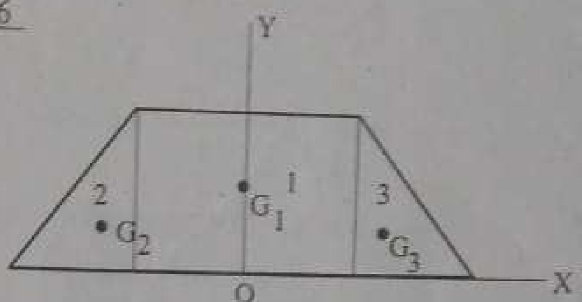

Fig. 3.38

When the axis is taken as shown in Fig. 3.38

$$\bar{x} = 0$$

Example 3.17

A semi circle area of 20 cm diameter was removed from a thin plate of 50×50 cm as shown in Fig. 3.39. It is desired to maintain the centroid of the plate at the original point. Determine the width of the material, shown shaded at the left, to be removed to achieve this.

Solution:

Since the plate is symmetrical about XX axis,

$$\bar{y} = \frac{50}{2} = 25 \text{ cm}$$

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2 - a_3 x_3}{a_1 - a_2 - a_3}$$

$$a_1 = 50 \times 50 = 2500 \text{ cm}^2$$

$$a_2 = 50 \times b = 50\,b \text{ cm}^2$$

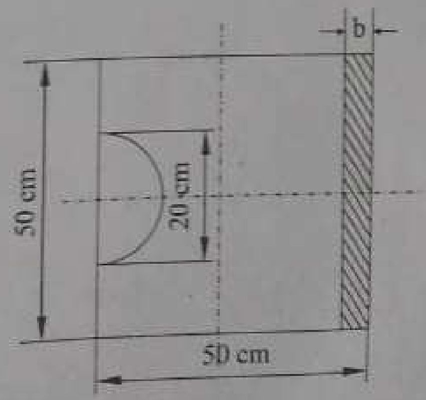

Fig. 3.39

$$a_3 = \frac{\pi r^2}{2} = \frac{\pi \times 10^2}{2} = 157\ cm^2$$

$$x_1 = \frac{50}{2} = 25\ cm$$

$$x_2 = \left(50 - \frac{b}{2}\right) cm$$

$$x_3 = \frac{4r}{3\pi} = \frac{4 \times 10}{3\pi} = 4.24\ cm$$

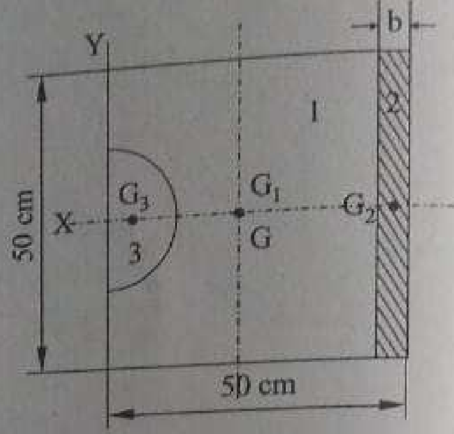

Fig. 3.40

$\bar{x} = x_1 = 25$ cm (required condition)

$$25 = \frac{2500 \times 25 - 50\,b \times \left(50 - \frac{b}{2}\right) - 157 \times 4.24}{2500 - 50\,b - 157}$$

$$25 = \frac{61834.32 - 2500\,b + 25\,b^2}{2343 - 50\,b}$$

$$58575 - 1250\,b = 61834.32 - 2500\,b + 25\,b^2$$

$$25\,b^2 - 1250\,b + 3259.32 = 0$$

$$b = 2.76\ cm$$

Example 3.18

Locate the centroid of the shaded area shown in Fig. 3.41.

Solution.

$$a_1 = 80 \times 40 = 3200\ mm^2$$

$$a_2 = \frac{1}{2} \times 40 \times 40 = 800\ mm^2$$

$$a_3 = \frac{\pi r^2}{4} = \frac{\pi \times 20^2}{4} = 314.16 mm^2$$

$$x_1 = \frac{80}{2} = 40\ mm$$

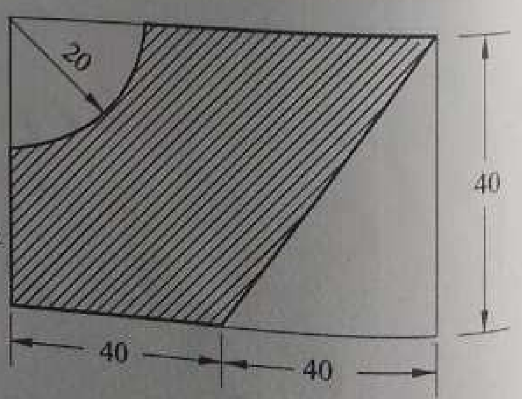

Fig. 3.41

$$x_2 = (80 - \frac{1}{3} \times 40) = 66.67 mm$$

$$x_3 = \frac{4r}{3\pi} = \frac{4 \times 20}{3\pi} = 8.49 \text{mm}$$

$$y_1 = \frac{40}{2} = 20 \text{ mm}$$

$$y_2 = \frac{1}{3} \times 40 = 13.33 \text{ mm}$$

$$y_3 = (40 - \frac{4 \times 20}{3\pi}) = 31.51 \text{mm}$$

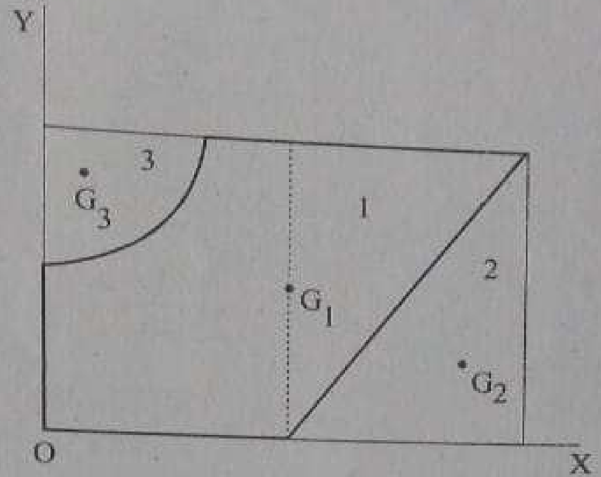

Fig.3.42

$$\bar{x} = \frac{a_1x_1 - a_2x_2 - a_3x_3}{a_1 - a_2 - a_3} = \frac{3200 \times 40 - 800 \times 66.67 - 314.16 \times 8.49}{3200 - 800 - 314.16}$$

$$= 34.52 \text{ mm}$$

$$\bar{y} = \frac{a_1y_1 - a_2y_2 - a_3y_3}{a_1 - a_2 - a_3} = \frac{3200 \times 20 - 800 \times 13.33 - 314.16 \times 31.51}{(3200 - 800 - 314.16)}$$

$$= 20.82 \text{ mm}$$

Example 3.19

Find the c.g. of the shaded area in Fig. 3.43

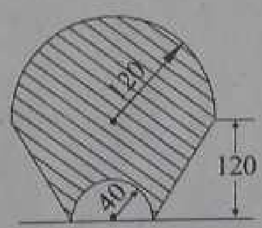

Fig. 3.43

Solution:

Since the shaded area is symmetrical about the Y axis, $\bar{x} = 0$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 - a_3y_3}{a_1 + a_2 + a_3}$$

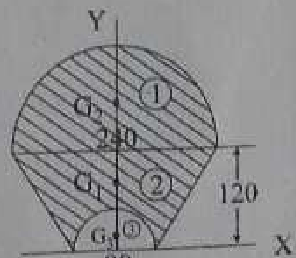

Fig. 3.44

$$a_1 = \left(\frac{240 + 80}{2}\right) \times 120$$

$$= 160 \times 120 = 19200 \text{ mm}^2$$

$$a_2 = \frac{\pi R^2}{2} = \frac{\pi \times 120^2}{2} = 22619.47 mm^2$$

$$a_3 = \frac{\pi r^2}{2} = \frac{\pi \times 40^2}{2} = 2513.27 mm^2$$

$$y_1 = \frac{h}{3}\left(\frac{b+2a}{b+a}\right)$$

$$= \frac{120}{3}\left[\frac{80+480}{80+240}\right]$$

$$= 70 \text{ mm}$$

$$y_2 = 120 + \frac{4R}{3\pi}$$

$$= 120 + \frac{4 \times 120}{3\pi}$$

$$= 170.93 \text{ mm}$$

$$y_3 = \frac{4r}{3\pi} = 16.98 \text{ mm}$$

$$\bar{y} = \frac{19200 \times 70 + 22619.47 \times 170.93 - 2513.27 \times 16.98}{19200 + 22619.47 - 2513.27}$$

$$= 131.47 \text{ mm}$$

Example 3.20.

Locate the centroid of the shaded area shown in Fig. 3.45

Solution.

$$a_1 = \frac{(160 + 200)}{2} \times 150 = 27000 \text{ mm}^2$$

$$a_2 = \frac{\pi \times 60^2}{2} = 5654.87 mm^2$$

$$y_1 = \left(\frac{2a+b}{a+b}\right)\frac{h}{3}$$

$$= \left(\frac{2 \times 160 + 200}{160 + 200}\right) \times \frac{150}{3}$$

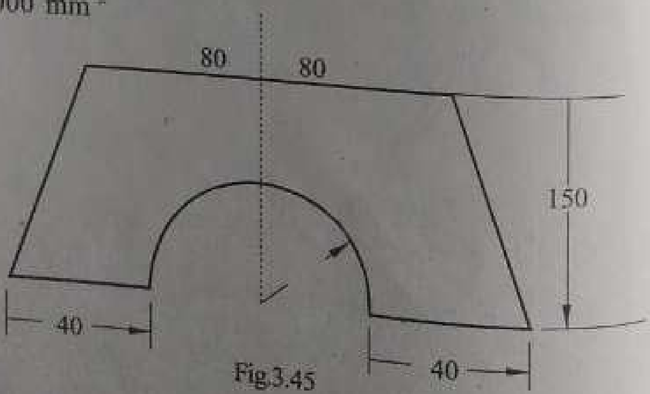

Fig.3.45

$= 72.22$ mm

$$y_2 = \frac{4r}{3\pi} = \frac{4 \times 60}{3\pi} = 25.46 \text{ mm}$$

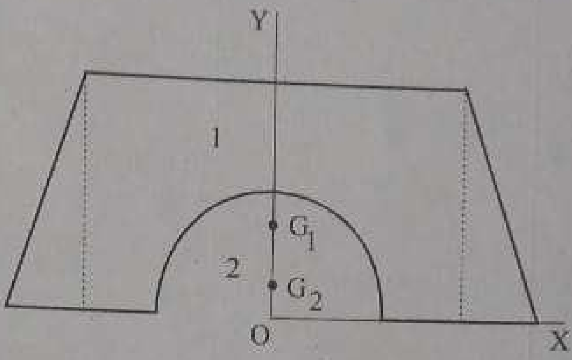

Fig. 3.46

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2}$$

$$= \frac{27000 \times 72.22 - 5654.87 \times 25.46}{27000 - 5654.87}$$

$= 84.61$ mm

$\bar{x} = 0$

Example 3.21

Locate the centroid of the shaded area as shown in Fig. 3.47

Solution.

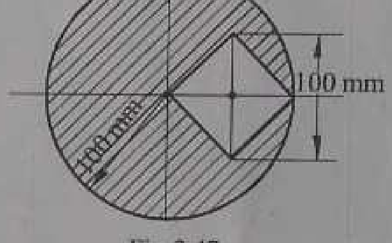

Fig. 3.47

$$a_1 = \pi r^2 = \pi \times 100^2$$

$$= 10000\,\pi \text{ mm}^2$$

$$a_2 = \frac{100^2}{2} = 5000 \text{ mm}^2$$

$x_1 = 100$ mm

$x_2 = 100 + 50 = 150$ mm

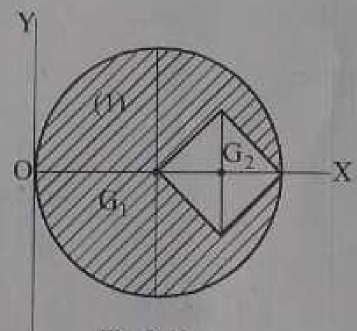

Fig. 3.48

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2}$$

$$= \frac{10000\pi \times 100 - 5000 \times 150}{10000\pi - 5000}$$

$= 90.54$ mm

$\bar{y} = 0$

Example 3.22

Locate the centroid of the shaded area shown in Fig. 3.49

Solution

$$a_1 = 100 \times 100 = 10000 \text{ mm}^2$$

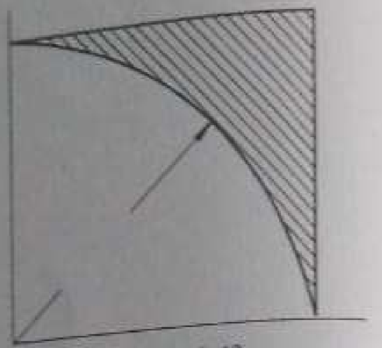

Fig. 3.49

$$a_2 = \frac{\pi r^2}{4} = \frac{\pi \times 100^2}{4} = 7853.98\,\text{mm}^2$$

$$x_1 = \frac{100}{2} = 50\text{ mm}$$

$$x_2 = \frac{4r}{3\pi} = \frac{4 \times 100}{3 \times \pi} = 42.44\text{ mm}$$

$$\bar{x} = \frac{a_1x_1 - a_2x_2}{a_1 - a_2} = \frac{10000 \times 50 - 7853.98 \times 42.44}{10000 - 7853.98}$$

$$= 77.67\text{ mm}$$

$$y_1 = \frac{100}{2} = 50\text{ mm}$$

$$y_2 = \frac{4r}{3\pi} = \frac{4 \times 100}{3\pi} = 42.44\text{ mm}$$

$$\bar{y} = \frac{a_1y_1 - a_2y_2}{a_1 - a_2}$$

$$= \frac{10000 \times 50 - 7853.98 \times 42.44}{10000 - 7853.98}$$

$$= 77.67\text{ mm}$$

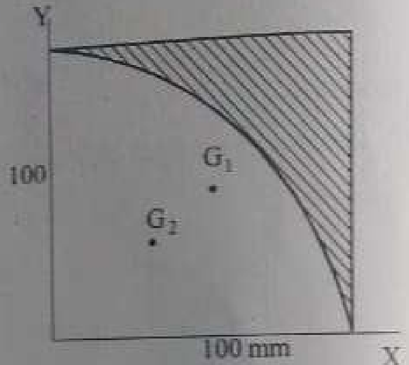

Fig. 3.50

Example 3.23

Locate the centroid of the shaded area shown in Fig. 3.51

Solution.

$$a_1 = 9 \times 9 = 81\text{ cm}^2$$

$$a_2 = a_3 = \frac{\pi r^2}{4} = \frac{\pi \times 9^2}{4}$$

$$= 63.62\text{ cm}^2$$

$$x_1 = \frac{9}{2} = 4.5\text{ cm}$$

$$x_2 = 9 + \frac{4r}{3\pi} = 9 + \frac{4 \times 9}{3\pi} = 12.82\text{cm}$$

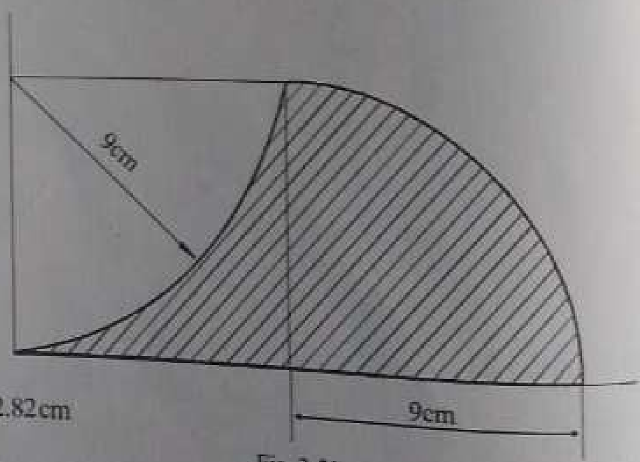

Fig. 3.51

$$x_3 = \frac{4r}{3\pi} = \frac{4 \times 9}{3\pi} = 3.82 \text{ cm}$$

$$y_1 = \frac{9}{2} = 4.5 \text{ cm}$$

$$y_2 = \frac{4r}{3\pi} = \frac{4 \times 9}{3\pi}$$

$$= 3.82 \text{ cm}$$

$$y_3 = r - \frac{4r}{3\pi} = 9 - \frac{4 \times 9}{3\pi}$$

$$= 5.18 \text{ cm}$$

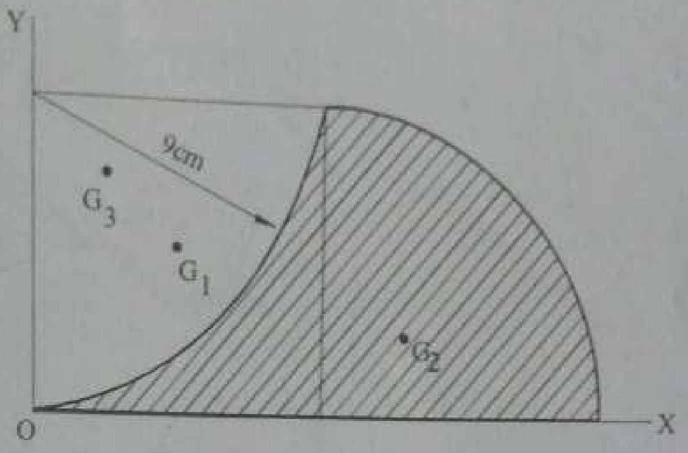

Fig. 3.52

$$\bar{x} = \frac{a_1x_1 + a_2x_2 - a_3x_3}{a_1 + a_2 - a_3} = \frac{81 \times 4.5 + 63.62 \times 12.82 - 63.62 \times 3.82}{81 + 63.62 - 63.62}$$

$$= 11.57 \text{ cm}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 - a_3y_3}{a_1 + a_2 - a_3} = \frac{81 \times 4.5 + 63.62 \times 3.82 - 63.62 \times 5.18}{81 + 63.62 - 63.62}$$

$$= 3.43 \text{ cm}$$

Example 3.24

Locate the centroid of the shaded area as shown in Fig. 3.53

Tabular method.

Section	a	x	y	ax	ay
1	40.50	3	-3	121.5	-121.5
2	63.62	3.82	3.82	243.03	243.03
3	40.50	-3	3	-121.5	121.5
4	81.00	-4.5	-4.5	-364.5	-364.5
5	-63.62	-5.18	-5.18	329.55	329.55
	162			208.08	208.08
	Σa			Σ(ax)	Σ(ay)

$$a_1 = \frac{1}{2} \times 9 \times 9 = 40.50\, cm^2$$

$$a_2 = \frac{\pi r^2}{4} = \frac{\pi \times 9^2}{4} = 63.62\, cm^2$$

$$a_3 = \frac{1}{2} \times 9 \times 9 = 40.50\, cm^2$$

$$a_4 = 9 \times 9 = 81\, cm^2$$

$$a_5 = -\frac{\pi r^2}{4} = -\frac{\pi \times 9^2}{4} = -63.62\, cm^2$$

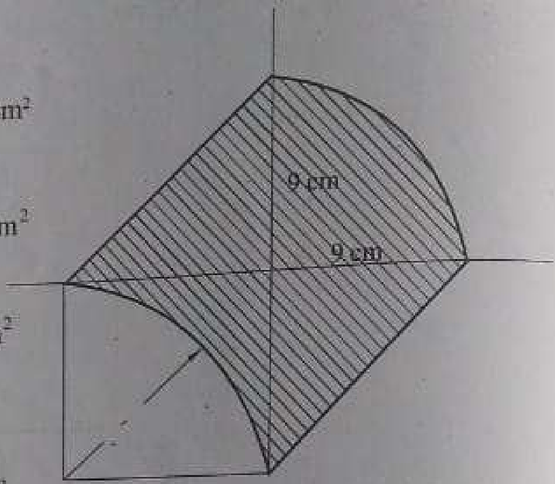

Fig. 3.53

$$x_1 = \frac{1}{3} \times 9 = 3\, cm$$

$$x_2 = \frac{4r}{3\pi} = \frac{4 \times 9}{3\pi} = 3.82\, cm$$

$$x_3 = -\frac{1}{3} \times 9 = -3\, cm$$

$$x_4 = -\frac{9}{2} = -4.5\, cm$$

$$x_5 = -\left[r - \frac{4r}{3\pi}\right] = -\left[9 - \frac{4 \times 9}{3\pi}\right] = -5.18$$

$$y_1 = -\frac{9}{3} = -3\, cm$$

$$y_2 = \frac{4r}{3\pi} = \frac{4 \times 9}{3\pi} = 3.82\, cm$$

$$y_3 = \frac{1}{3} \times 9 = 3\, cm$$

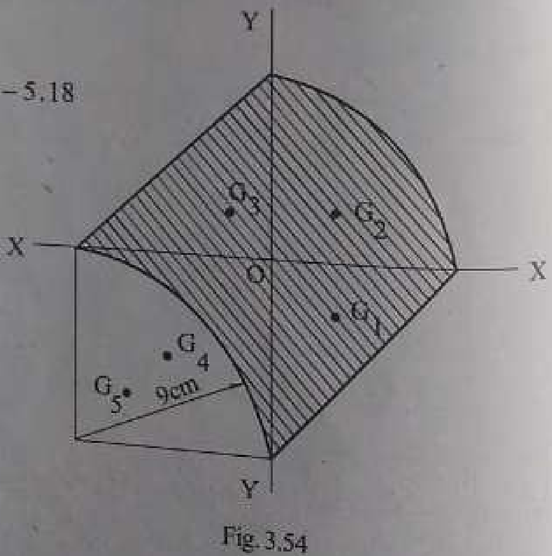

Fig. 3.54

$$y_4 = -\frac{9}{2} = -4.5 \text{ cm}$$

$$y_5 = -\left[r - \frac{4r}{3\pi}\right] = -\left[9 - \frac{4 \times 9}{3\pi}\right] = -5.18 \text{ cm}$$

$$\bar{x} = \frac{\Sigma(ax)}{\Sigma a} = \frac{208.08}{162} = 1.28 \text{ cm}$$

$$\bar{y} = \frac{\Sigma(ay)}{\Sigma a} = \frac{208.08}{162} = 1.28 \text{ cm}$$

Example 3.25

Locate the centroid of the shaded area as shown in fig. 3.55

Solution.

Consider a vertical strip of width dx at a distance x from the Y axis. Area of the element

$d_A = y\,dx = kx^2\,dx$

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^{25} x\,kx^2\,dx}{\int_0^{25} kx^2\,dx} = \frac{k\left[\frac{x^4}{4}\right]_0^{25}}{k\left[\frac{x^3}{3}\right]_0^{25}}$$

$$= \frac{\frac{25^4}{4}}{\frac{25^3}{3}} = \frac{3}{4} \times 25 = 18.75 \text{ cm}$$

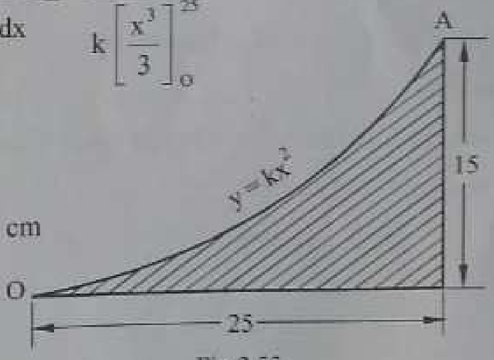

Fig. 3.55

$$\bar{y} = \frac{\int y\,dA}{\int dA}$$ where y is the distance of centroid of elemental strip from the X axis. Since the elemental strip is a rectangle of height y, the value of y in the expression for $\bar{y}$ is $\frac{y}{2}$, which is equal to $\frac{kx^2}{2}$ and $dA = y\,dx = kx^2\,dx$.

$$\bar{y} = \frac{\int_0^{25} \frac{kx^2}{2} kx^2 \, dx}{\int_0^{25} kx^2 \, dx} = \frac{\frac{k^2}{2}\left[\frac{x^5}{5}\right]_0^{25}}{k\left[\frac{x^3}{3}\right]_0^{25}}$$

$$= \frac{3}{10} k. 25^2 = 187.5 \text{ k.}$$

At A, $x = 25$ and $y = 15$

$$y = kx^2$$

$$15 = k \times 25^2$$

$$\therefore k = \frac{15}{25^2}$$

$$\therefore \bar{y} = 187.5 \times \frac{15}{25^2}$$

$$\bar{y} = 4.5 \text{ cm}$$

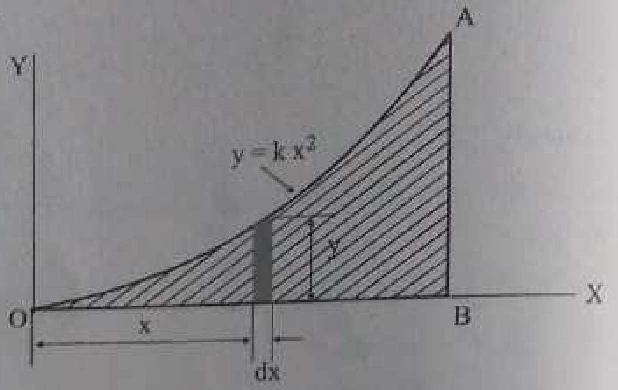

Fig. 3.56

Example 3.26

Locate the centroid of the shaded area shown in Fig. 3.57

Solution.

Consider a vertical strip of height h and width dx.

The area of the element,

$$dA = h\,dx = (2x - \frac{x^2}{2})\,dx$$

$$\bar{x} = \frac{\int x \, dA}{\int dA}$$

$y = \frac{x^2}{2}$. At A, $x = 4$ and $y = 8$

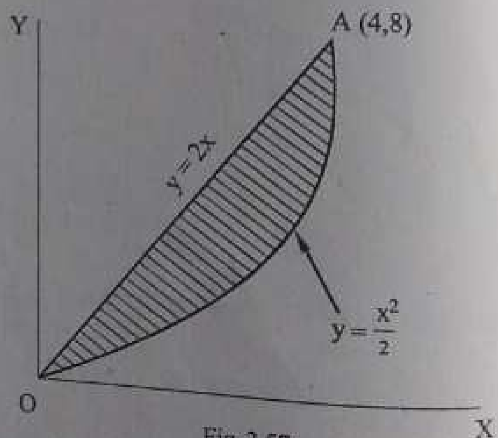

Fig. 3.57

$$\bar{x} = \frac{\int_0^4 x\,(2x - \frac{x^2}{2})\,dx}{\int_0^4 (2x - \frac{x^2}{2})\,dx}$$

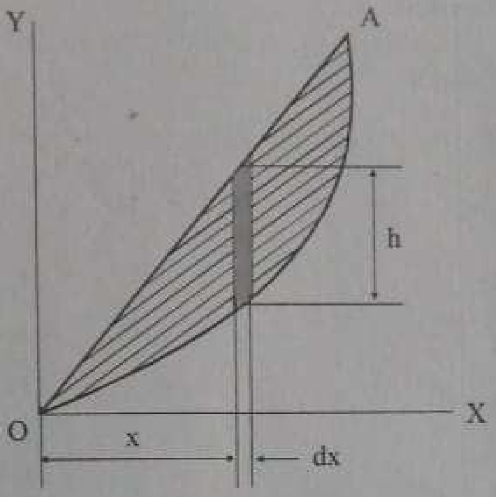

Fig. 3.58

$$= \frac{2\left[\frac{x^3}{3}\right]_0^4 - \frac{1}{2}\left[\frac{x^4}{4}\right]_0^4}{2\left[\frac{x^2}{2}\right]_0^4 - \frac{1}{2}\left[\frac{x^3}{3}\right]_0^4}$$

$$= \frac{\frac{2 \times 4^3}{3} - \frac{1}{2} \times \frac{4^4}{4}}{2 \times \frac{4^2}{2} - \frac{1}{2} \times \frac{4^3}{3}}$$

$$\bar{x} = 2$$

Consider a horizontal strip of thickness dy and length l as shown in Fig. 3.59, at a distance y from the x axis.

area of the elemental strip,

$$dA = l \times dy.$$

$$= (\sqrt{2y} - \frac{y}{2})\,dy$$

$$\bar{y} = \frac{\int y.dA}{\int dA} = \frac{\int_0^8 (\sqrt{2} \times y^{\frac{3}{2}} - \frac{1}{2}y^2)dy}{\int_0^8 (\sqrt{2} \times y^{\frac{1}{2}} - \times\frac{1}{2}y)dy}$$

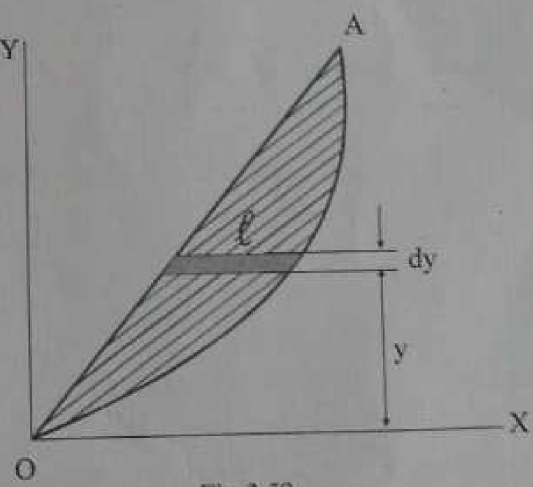

Fig. 3.59

$$\bar{y} = \frac{\left[\sqrt{2}\,y^{\frac{5}{2}} \times \frac{2}{5} - \frac{1}{2} \times y^3 \times \frac{1}{3}\right]_0^8}{\left[\sqrt{2}\,y^{\frac{3}{2}} \times \frac{2}{3} - \frac{1}{2}y^2 \times \frac{1}{2}\right]_0^8} = \frac{\frac{2\sqrt{2}}{5} \times 8^{\frac{5}{2}} - \frac{1}{6} \times 8^3}{2\frac{\sqrt{2}}{3} \times 8^{\frac{3}{2}} - \frac{1}{4} \times 8^2} = \frac{17.07}{5.33} = 3.2$$

Example 3.27

Determine the centroid of the quadrant of the ellipse shown in the Fig. 3.60

Solution :

The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

$$x=\frac{a}{b}\sqrt{b^2-y^2}$$

To calculate $\bar{x}$

Consider an elemental strip of thickness dx at a distance x from the y axis. Let y be the height of this element.

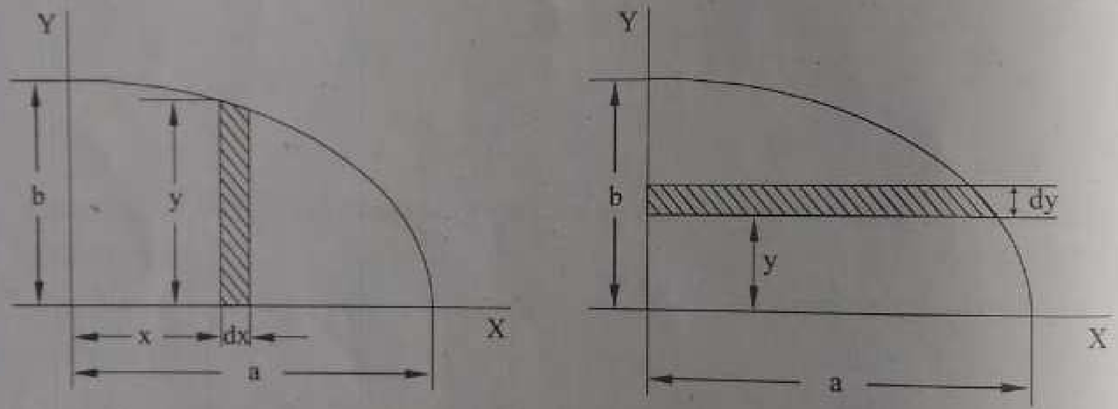

Fig. 3.60

$$\bar{x}=\frac{\int x\, dA}{\int dA}$$

$$\int dA=\int_0^a y\, dx$$

$$=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\frac{\pi a^2}{4}=\frac{\pi ab}{4} \qquad \left(\int_0^a \sqrt{a^2-x^2}\,dx=\frac{\pi a^2}{4}\right)$$

$$= \int_0^a x \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\; x\,dx$$

Put $a^2 - x^2 = t^2$

Differentiating both sides,

$$-2x\,dx = 2t\,dt$$

$$x\,dx = -\,t\,dt$$

$$\int x\,dA = \frac{b}{a}\int_0^a t\left[-t\,dt\right] = -\frac{b}{a}\left[\frac{t^3}{3}\right]_0^a$$

$$= -\frac{b}{3a}\left[\left(a^2 - x^2\right)^{\frac{3}{2}}\right]_0^a$$

$$= -\frac{b}{3a}\left[\left(a^2 - a^2\right)^{\frac{3}{2}} - \left(a^2 - 0\right)^{\frac{3}{2}}\right] = \frac{b}{3a}\left(0 - a^3\right)$$

$$= \frac{ba^2}{3} \qquad \therefore \bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\frac{ba^2}{3}}{\frac{\pi ab}{4}}$$

$$\bar{x} = \frac{4a}{3\pi}$$

Similarly by considering a horizontal strip of thickness dy at a distance y from the x axis, it can be proved that $\bar{y} = \dfrac{4b}{3\pi}$

3.3. Theorems of Pappus - Guldinus.

The theorems of Pappus - Guldinus were first set forth by Pappu about 300 A.D and then restated by the swiss mathematician Guldinus about 1640. These theorems offer a simple way for computing the area of surface of revolution and the volume of bodies of revolution.

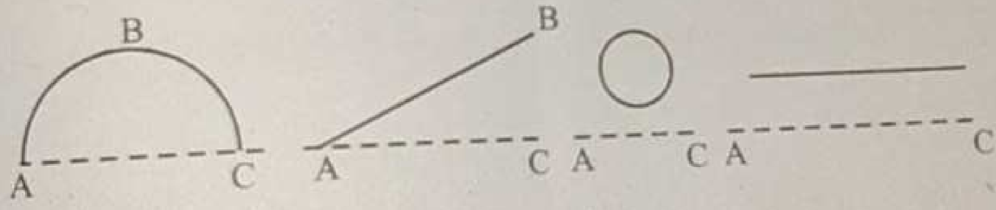

Fig. 3.61

A surface of revolution is a surface which may be generated by rotating a plane curve about a fixed axis. The surface of a sphere is obtained by rotating a semicircular Arc ABC about the axis AC, the surface of a cone by rotating inclined line AB about axis AC, the surface of a ring by rotating the circumference of a circle about the axis AC, the surface of a cylinder by rotating a horizontal line about the axis AC as shown in Fig 3.61.

A body of revolution is the body which is generated by rotating a plane area about a fixed axis. A solid sphere is generated by rotating a semicircular area, a cone by rotating a triangular area and a cylinder by rotating a rectangular area about axis AB shown in Fig. 3.62.

Fig. 3.62

Theorem 1.

The area of surface generated by revolving a plane curve about a non-intersecting axis in the plane of the curve is equal to the product of length of curve and the distance travelled by the centroid of the curve while the surface is being generated.

Proof.

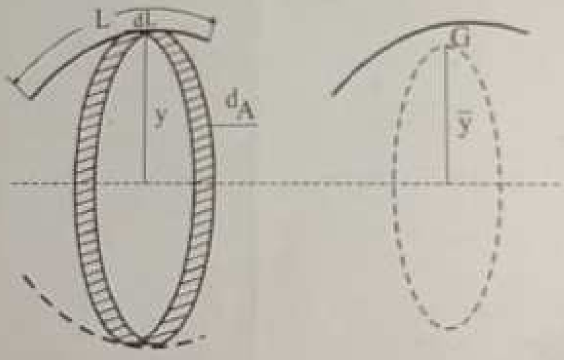

Fig. 3.63

Consider an element of length dL of the curve of length L which is revolved about the X axis. The area generated by the element is equal to $2\pi y\, dL$, where y is the distance of element from x axis. Therefore the entire area generated by the curve, $A = \int 2\pi y\, dL$

$= 2\pi \int (y\, dL) = 2\pi \bar{y} L$

$2\pi \bar{y}$ is the distance travelled by the centroid of curve of length L.

Theorem II

The volume of a body generated by revolving a plane area about a non-intersecting axis in the plane of the area is equal to the product of area and the distance travelled by the centroid of the plane area while the body is being generated.

Proof.

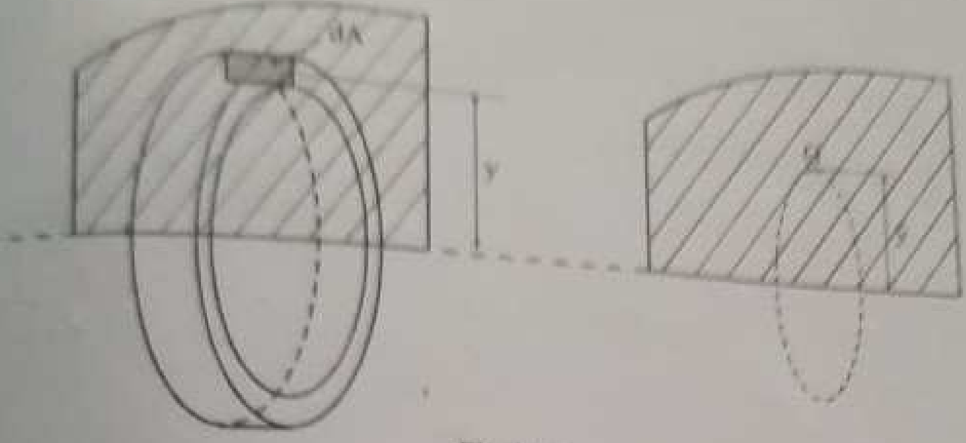

Fig. 3.64

Consider an element dA of the area A which is revolved about the x axis. The volume dv generated by the element dA in one revolution is equal to $2\pi y dA$. Therefore the entire volume generated by A, $V = \int 2\pi y dA = 2\pi \int y dA$

$= 2\pi \bar{y} A$. Where $2\pi \bar{y}$ is the distance travelled by the centroid of area A.

Example 3.28.

Obtain the expression for the area of surface generated by rotation of a horizontal axis at a distance r from the line.

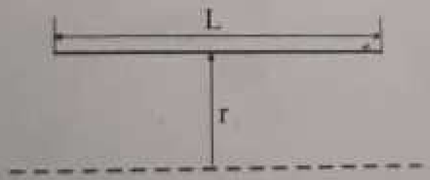

Fig. 3.65

Solution

Length of line = L. distance of centroid of line from the axis of rotation, r. Distance travelled by G in one revolution is $2\pi \bar{y} = 2\pi r$.

Surface area generated $= L \times 2\pi r$

$= 2\pi r \times L$

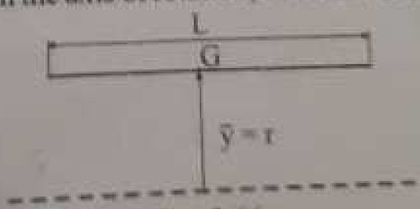

Fig. 3.66

Example 3.29

Obtain the expression for the area of surface generated by rotation of a semicircular arc of radius r about an axis passing through its end points.

distance of centroid from the axis is $\frac{2r}{\pi}$ Hollow

Length of arc ABC is πr. distance travelled by the centroid in one revolution is $2\pi \times \frac{2r}{\pi} = 4r$

area of surface generated $= \pi r \times 4r$

$= 4\pi r^2$

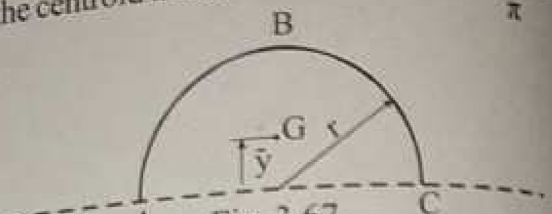

Fig. 3.67

Example 3.30

Obtain an expression for the area of surface generated when a line of length L revolve about an axis. One end of the line is touching the axis and the other end is at a distance r from the axis.

Solution.

Length of line = L

Distance of centroid G from the axis is $\frac{r}{2}$. Distance travelled by the centroid in one revolution is $2\pi\frac{r}{2} = \pi r$

∴ area of surface generated $= L \times \pi r$

$= \pi r L$

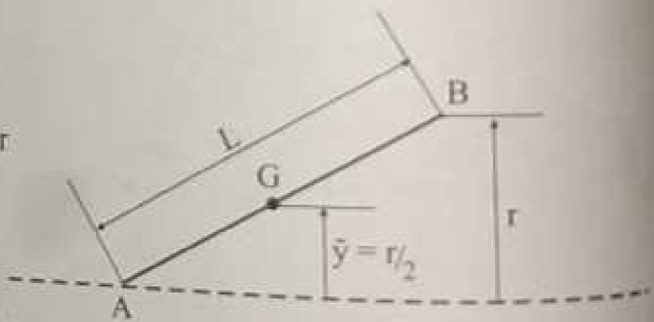

Fig. 3.68

Example 3.31

Calculate the surface area obtained by revolving the line ABC as shown in Fig. 3.69 about (i) X axis (ii) Y axis.

Solution.

Length of line $= 18 + 9 + \sqrt{18^2 + 9^2}$

$= 18 + 9 + 20.1 = 47.1$ cm

Distance of centroid from Y axis, $\bar{x} = \frac{L_1x_1 + L_2x_2 + L_3x_3}{L_1 + L_2 + L_3}$

$= \frac{18 \times 0 + 9 \times 4.5 + 20.1 \times 4.5}{47.1}$

$= 2.78$ cm

Distance travelled by centroid in one revolution about Y axis is

$2\pi \times 2.78$ cm.

Area of surface generated $= 47.1 \times 2\pi \times 2.78 = 822.71 \text{ cm}^2$

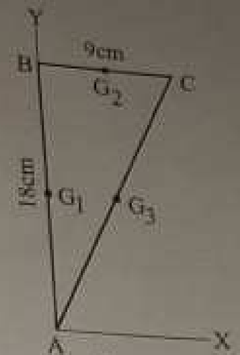

Fig. 3.69

Distance of centroid from x axis $\bar{y} = \frac{y_1L_1 + y_2L_2 + y_3L_3}{L_1 + L_2 + L_3}$

$= \frac{9 \times 18 + 18 \times 9 + 20.1 \times 9}{47.1}$

$= 10.72$ cm

Distance travelled by centroid in one revolution about X axis is $2\pi \times 10.72$ cm.

Area of surface generated $= 47.1 \times 2\pi \times 10.72 = 3172.46 \text{ cm}^2$

Example 3.32

Obtain an expression for the volume of body generated by revolution of a rectagular area. The side L of the rectangle is in tough with the axis of rotation and the other side is of length r.

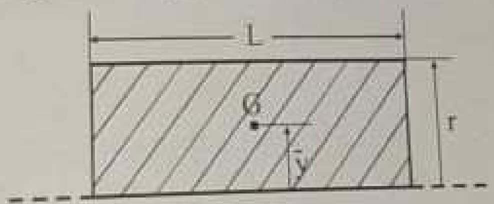

Fig. 3.70

Area of rectangle $= L \times r$. Distance of centroid from the axis, $\bar{y} = \frac{r}{2}$. Distance travelled by the centroid in one revolution is $2\pi\bar{y} = 2\pi\frac{r}{2} = \pi r$. Volume of body generated $= L \times r \times \pi r = \pi r^2 L$.

Example 3.33

Obtain an expression for the volume of body generated by revolution of a semicircular area when its base is in a touch with the axis of rotation.

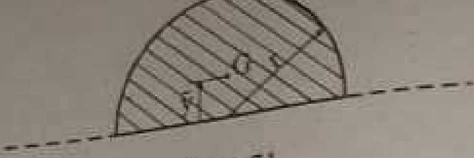

Fig. 3.71

Area of semicircle = $\frac{\pi r^2}{2}$. Distance of centroid from the base is $\frac{4r}{3\pi}$. Distance travelled by the centroid in one revolution is $2\pi \times \frac{4r}{3\pi} = \frac{8}{3}r$. Therefore volume of body generated $= \frac{\pi r^2}{2} \times \frac{8}{3} r = \frac{4}{3} \pi r^3$.

Example 3.34

Obtain an expression for the volume of body generated by revolution of a triangle when its base h is in touch with the axis of rotation and the length of other side is r.

Area of triangle = $\frac{1}{2} \times h \times r$.

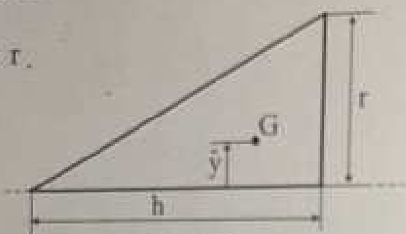

Fig. 3.72

Distance of centroid from base is $\frac{1}{3} \times r$. Distance travelled by the centroid in one revolution is $2\pi \times \frac{1}{3} r = \frac{2}{3} \pi r$

Therefore, volume of body generated $\frac{1}{2} \times h \times r \times \frac{2}{3} \pi r = \frac{1}{3} \pi r^2 h$.

3.4. Moment of inertia

Moment of Inertia of an area is a purely mathematical term which gives a quantitative estimate of the relative distribution of area with respect to some reference axis. If r is the distance of an elemental area, dA, from a reference axis AB, then the sum of the terms $\Sigma r^2 dA$, to cover the entire area is called moment of inertia of the area about the reference axis AB and is denoted by I_{AB}.

$$I_{AB} = \Sigma r^2 dA = \int r^2 dA$$

The product $r \times dA$ is the first moment of elemental area about the reference axis. The second moment of elemental area about the reference axis is $r^2 dA$. Thus the moment of inertia of an area is the integral of second moment of an elemental area about a

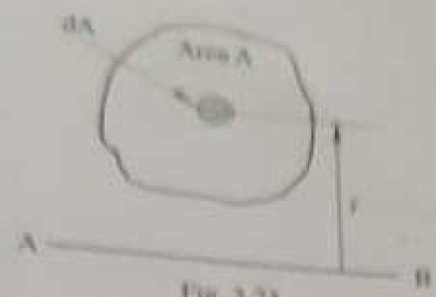

Fig. 3.73

reference axis. The term second moment of area is more proper than moment of inertia because the term moment of inertia should be used when the integral of mass is taken. Generally the moment of inertia of area is called area moment of inertia or second moment of area and moment of inertia of physical body is called mass moment of inertia The axis passing through the centroid is called centroidal axis and the moment of inertia about the centroidal axis is denoted by I_G .

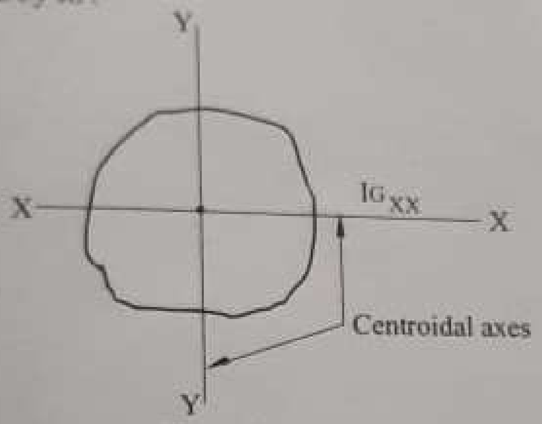

Fig. 3.74

3.5. Radius of gyration of an area.

Consider an area A which has a moment of inertia I with respect to a reference axisAB. Let us assume that this area is compressed to a thin strip parallel to the axis AB. For this strip to have the same moment of inertia I, with respect to the same reference axis AB, the strip should be placed at a distance k from the axis AB such that $I = Ak^2$. $k = \sqrt{\frac{I}{A}}$ is called radius of gyration of the area with respect to the given axis AB.

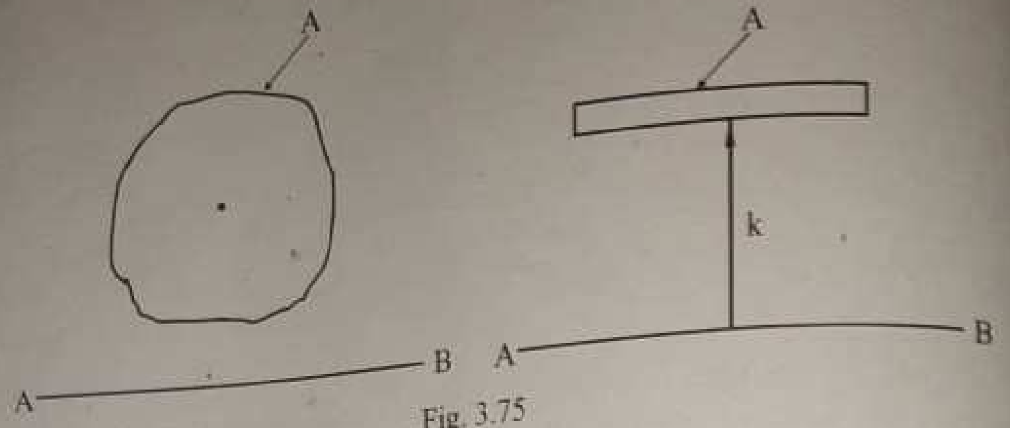

Fig. 3.75

3.6. Perpendicular axis theorem.

If I_{XX} and I_{YY} are the moment of inertia of an area A about mutually perpendicular axis XX and YY, in the plane of the area, then the moment of inertia of the area about the ZZ axis which is perpendicular to XX and YY axis and passing through the point of intersection of XX and YY axis is given by $I_{ZZ} = I_{XX} + I_{YY}$

Proof:

Consider a plane area A. Let XX and YY be the two mutually perpendicular axes in the plane of the area, intersecting at O. Let ZZ be an axis through O and perpendicular

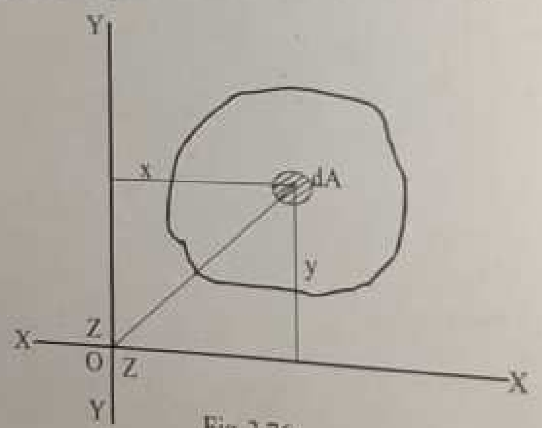

Fig. 3.76

to the plane of area A. Consider an elemental area dA at a distance y from the XX axis, x from the YY axis and r from the ZZ axis, then $I_{ZZ} = \int r^2 dA = \int (x^2+y^2)\, dA$

$$= \int x^2 dA + \int y^2 dA$$

$$= I_{XX} + I_{YY} = I_{YY} + I_{XX}$$

$$I_{ZZ} = I_{XX} + I_{YY}.$$

3.7. Polar moment of inertia

Moment of inertia about ZZ axis which is perpendicular to XX and YY axis and passing through the point of intersection of XX and YY axis is called Polar moment of inertia. According to perpendicular axis theorem, $I_{zz} = I_{xx} + I_{yy}$. Thus polar moment of inertia $I_{zz} = I_{xx} + I_{yy}$

3.8. Parallel axis theorem.

It is a transfer theorem which is used to transfer moment of inertia from one axis to another axis. These two axes should be parallel to each other and one of these axes must be a centroidal axis.

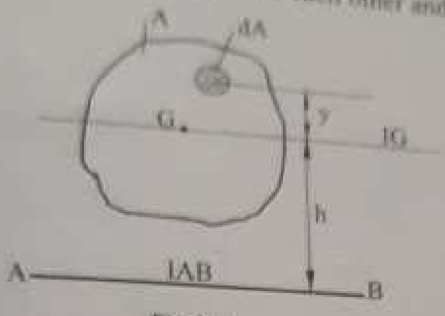

Fig 3.77

It states that, if IG is the moment of inertia of a plane lamina of area A, about its centroidal axis in the plane of the lamina, then the moment of inertia about any axis AB which is parallel to the centroidal axis and at a distance 'h' from the centroidal axis is given by

$$I_{AB} = I_G + A h^2$$

Proof.

Consider an elemental area dA at a distance y from the centroidal axis. The first moment of elemental area about the axis AB as shown in Fig.3.77 is $dA\,(y + h)$. Second moment of elemental area about the axis AB is $dA\,(y + h)^2$. The second moment of the area about the axis AB is $\int dA\,(y + h)^2$

$$I_{AB} = \int dA\,(y + h)^2 = \int dA\,(y^2 + h^2 + 2\,hy)$$

$$= \int y^2 dA + \int h^2 dA + \int 2\,h\,y\,dA$$

$$= I_G + h^2 \int dA + 2h \int y\,dA = I_G + h^2 A + 2\,h\,(A\bar{y})$$

$$I_{AB} = I_G + Ah^2$$

$\bar{y} = 0$, because it is the distance of centroid G from the axis from which y is measured. In Fig 3.77, y is measured from the centroidal axis itself.

isothermal, heat ... possible, ...

and for rejecting heat to the surroundings ... heat from the space to be refrigerated, it is possible to maintain ... refrigerated.

Example 3.35

Calculate the moment of inertia of a rectangular cross section about the centroidal axis and also about its base AB

Solution.

Consider an element of thickness dy at a distance y from the X axis.

Area of the element, $dA = b.dy$.

Second moment of this elemental area about the X axis, $d\,I_{XX} = y^2\,d\,A$

$$= y^2\,b.\,d\,y$$

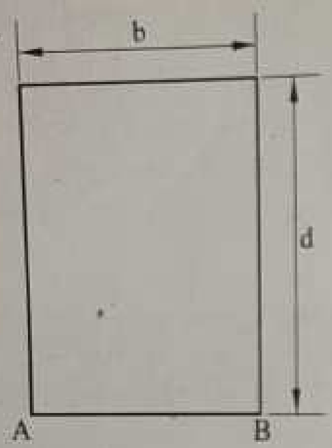

Fig 3.78

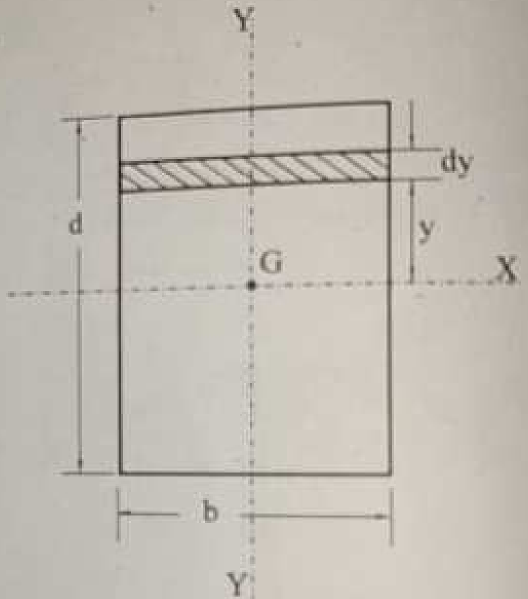

Fig. 3.79

$$\therefore I_{XX} = \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 b\,dy$$

$$= b.\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$= \frac{b}{3}\left[\left(\frac{d}{2}\right)^3 - \left(-\frac{d}{2}\right)^3\right]$$

$$= \frac{b}{3}\left[\frac{d^3}{8} + \frac{d^3}{8}\right]$$

$$= \frac{b}{3}\,\frac{d^3}{4}$$

$$I_{XX} = \frac{b\,d^3}{12}$$

Consider an element of thickness dx at a distance x from the Y axis.

Area of the element $dA = d \times dx$.

Second moment of this elemental area about the Y axis,

$$d\,I_{YY} = x^2\,d\,A$$

$$= x^2\; d \times dx$$

$$I_{YY} = \int_{\frac{b}{2}}^{\frac{b}{2}} x^2\,d\,dx$$

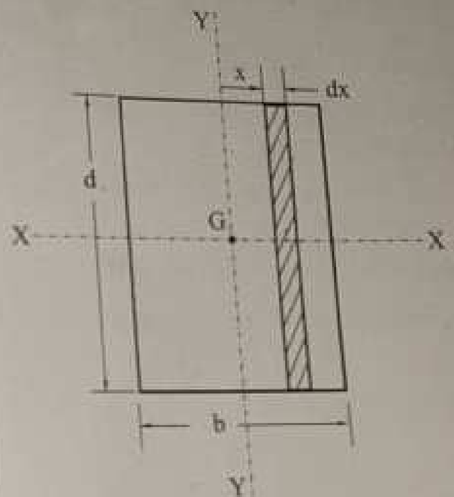

Fig. 3.80

$$= d\left[\frac{x^3}{3}\right]_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$= \frac{d}{3}\left[\left(\frac{b}{2}\right)^3 - \left(\frac{-b}{2}\right)^3\right]$$

$$= \frac{d}{3}\left[\frac{b^3}{8} + \frac{b^3}{8}\right]$$

$$= \frac{d}{3}\,\frac{b^3}{4} = \frac{d\,b^3}{12}$$

$$I_{YY} = \frac{d\,b^3}{12}$$

Moment of inertia of a rectangular lamina about its centroidal axis $= \frac{1}{12} \times$ length of side parallel to the axis $\times$ (length of other side $)^3$.

Moment of inertia about the base can be calculated using parallel axis theorem

$$I_{AB} = I_G + A h^2$$

$$= \frac{b d^3}{12} + b d \times \left(\frac{d}{2}\right)^2$$

$$= \frac{b d^3}{12} + \frac{b d^3}{4}$$

$$I_{AB} = \frac{bd^3}{3}$$

Moment of inertia of a rectangular lamina about its base is $\frac{bd^3}{3}$

Example 3.36

Determine the moment of inertia of a triangle about its centroidal axis, parallel to its base.

Consider an element of thickness dy at a distance y from the vertex R. Let the length of this element be b.

Area of the element, dA = b dy.

Second moment of this elemental area about the axis through the vertex R,

$$dI_R = y^2 (b\, dy)$$

$$I_R = \int_0^H b\, y^2\, dy$$

$$\left(\frac{b}{B} = \frac{y}{H}\right) \therefore b = \frac{B}{H} \times y$$

$$I_R = \int_0^H \frac{B}{H}\, y\, y^2\, dy$$

$$= \frac{B}{H} \left[\frac{y^4}{4}\right]_0^H$$

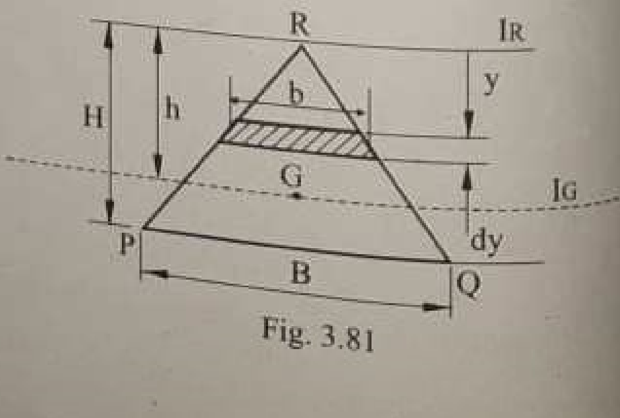

Fig. 3.81

$$= \frac{B}{H} \cdot \frac{H^4}{4}$$

$$I_R = \frac{BH^3}{4}$$

Using parallel axis theorem,

$$I_R = I_G + Ah^2,$$

$$I_G = I_R - Ah^2$$

$$= \frac{BH^3}{4} - \frac{1}{2} \times B \times H \times (\frac{2}{3} H)^2$$

$$= \frac{BH^3}{4} - \frac{2}{9} \times BH^3$$

$$I_G = \frac{BH^3}{36}$$

Again using parallel axis theorem,

$$I_{PQ} = I_G + Ah^2$$

$$= \frac{BH^3}{36} + \frac{1}{2} \times B \times H \times (\frac{H}{3})^2$$

$$= \frac{BH^3}{36} + \frac{BH^3}{18}$$

$$I_{PQ} = \frac{BH^3}{12}$$

Moment of inertia of a triangular lamina about its base is $\frac{BH^3}{12}$.

Moment of inertia of a triangular lamina about the centroidal axis, parallel to the base is $\frac{BH^3}{36}$.

Moment of inertia of a triangular lamina about an axis through the vertex and parallel to the base is $\frac{BH^3}{4}$.

Example 3.37

Determine the moment of inertia of a circular lamina about its centroidal axes.

Solution.

Consider an elemental area as shown in Fig 3.82. The area of the element, $dA = r\,d\theta \times dr$

Second moment of this elemental area about the XX axis, $dI_{xx} = y^2\,dA$

$$\therefore I_{XX} = \int y^2\, r\, d\theta\, dr$$

$$= \int_0^{2\pi}\int_0^{R} (r\sin\theta)^2\, r\, d\theta\ dr$$

$$= \int_0^{2\pi}\int_0^{R} r^3 \sin^2\theta\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^R \sin^2\theta\, d\theta$$

$$= \frac{R^4}{4}\int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \frac{R^4}{4}\int_0^{2\pi}\left[\frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{R^4}{4\times 2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{R^4}{8}\,[2\pi - \theta - (0-0)]$$

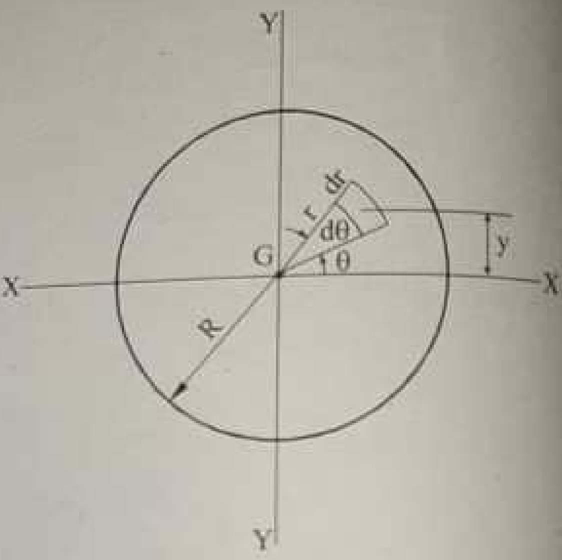

Fig. 3.82

$$= \frac{\pi R^4}{4}$$

$I_{XX} = \frac{\pi R^4}{4}$, where D is the diameter of the circle. Because of symmetry of circular area, $I_{YY} = I_{XX} = \frac{\pi R^4}{4}$

Example 3.38

Determine the moment of inertia of a semicircular lamina about its centroidal axes.

Consider an elemental area as shown in Fig 3.83

Area of the element $dA = r\, d\theta\, dr$

Second moment of this elemental area about the base AB, $d I_{AB} = \int y^2\, dA$

$$y = r \sin\theta$$

$$I_{AB} = \int_0^{\pi} \int_0^{R} (r \sin\theta)^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{R} r^3 \sin^2 dr\, d\theta$$

$$= \frac{R^4}{4} \int_0^{\pi} \sin^2\theta\, d\theta$$

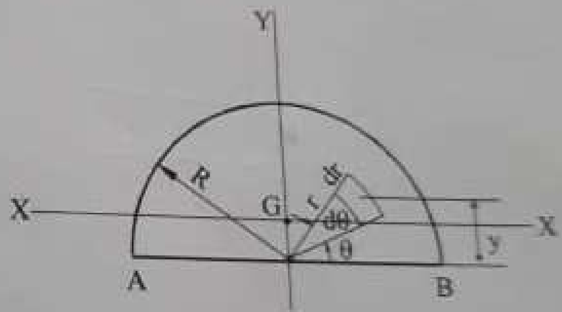

Fig.3.83

$$= \frac{R^4}{4} \int_0^{\pi} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta$$

$$= \frac{R^4}{4 \times 2} \left[\theta - 2 \times \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

Same method but 0 to π

$$= \frac{R^4}{8}\left[(\pi - 0) - (0 - 0)\right]$$

$$= \frac{\pi R^4}{8}$$

$$= \frac{1}{2} \times \frac{\pi R^4}{4}$$

Moment of inertia of a semicircular lamina about its base is $\boxed{\dfrac{\pi R^4}{8}}$

Using parallel axis theorem.

$$I_{AB} = I_G + Ah^2$$

$$I_{XX} = I_{AB} - Ah^2$$

$$= \frac{\pi R^4}{8} - \frac{\pi R^2}{2} \times \left(\frac{4R}{3\pi}\right)^2$$

$$= \frac{\pi R^4}{8} - \frac{R^4 \times 8}{9\pi}$$

$$= R^4\left[\frac{\pi}{8} - \frac{8}{9\pi}\right]$$

$$\boxed{I_{XX} = 0.11 R^4}$$

Moment of inertia of a semicircular lamina about its centroidal axis, parallel to the base is $0.11\ R^4$

Moment of inertia about the centroidal axis, perpendicular to the base.

Consider an elemental area as shown in Fig 3.83

Area of the element, $dA = r\, d\theta\, dr$

$$x = r \cos\theta$$

Second moment of elemental area about YY axis, $d\, I_{YY} = x^2\, dA$

$$\therefore I_{YY} = \int_0^{\pi}\int_0^{R} (r\cos\theta)^2\, r\, d\theta\, dr$$

i) ii) Lona- iii) Expansion

$$= \frac{R^4}{4}\int_0^{\pi} \cos^2\theta \, d\theta$$

$$= \frac{R^4}{4}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi}$$

$$= \frac{R^4}{4}\left[\frac{1}{2}\pi + 0\right]$$

$$= \frac{\pi R^4}{8} = \frac{1}{2}\times\frac{\pi R^4}{4}$$

Moment of inertia of a semicircular lamina about the centroidal axis, perpendicular to the base is $\frac{\pi R^4}{8}$

Example 3.39

Determine the moment of inertia of one quarter of a circle about its centroidal axes.

Consider an elemental area as shown in Fig.3.84. Area of the element, $dA = r\,d\theta\,dr$

$y = r\sin\theta$

Second moment of elemental area about the axis OA,

$$dI_{OA} = \int y^2 dA$$

$$I_{OA} = \int_0^{\frac{\pi}{2}}\int_0^{R} (r\sin\theta)^2\, r\,d\theta\;dr$$

$$= \frac{R^4}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

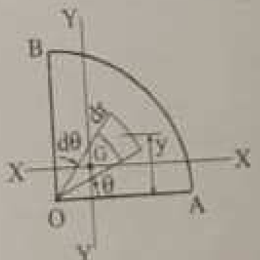

Fig. 3.84

$$= \frac{R^4}{4}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{R^4}{4}\left[\frac{1}{2} \times \frac{\pi}{2} - 0\right]$$

$$= \frac{\pi R^4}{16} = \frac{1}{2} \times \frac{\pi R^4}{8}$$

Because of symmetry, $I_{OA} = I_{OB} = \frac{\pi R^4}{16}$

Using parallel axis theorem,

$$I_{OA} = I_{XX} + Ah^2$$

$$I_{XX} = I_{OA} - Ah^2$$

$$= \frac{\pi R}{16} - \frac{\pi R^2}{4} \times \left[\frac{4R}{3\pi}\right]^2$$

$$= R^4\left[\frac{\pi}{16} - \frac{4}{9\pi}\right]$$

$$= 0.055\ R^4$$

Because of symmetry, $I_{XX} = I_{YY} = 0.055\ R^4$

Example 3.40

Calculate the moment of inertia of the angle section having the dimensions shown in Fig.3.85 about X and Y axis shown.

Solution.

$$A_1 = 10 \times 2 = 20\,\text{cm}^2$$

i) ii) Lona- iii) Expansion

$A_2 = 8 \times 2 = 16\ cm^2$

$x_1 = \frac{10}{2} = 5\ cm$

$x_2 = \frac{2}{1} = 1\ cm$

$y_1 = \frac{2}{1} = 1\ cm$

$y_2 = 2 + \frac{8}{2} = 6\ cm.$

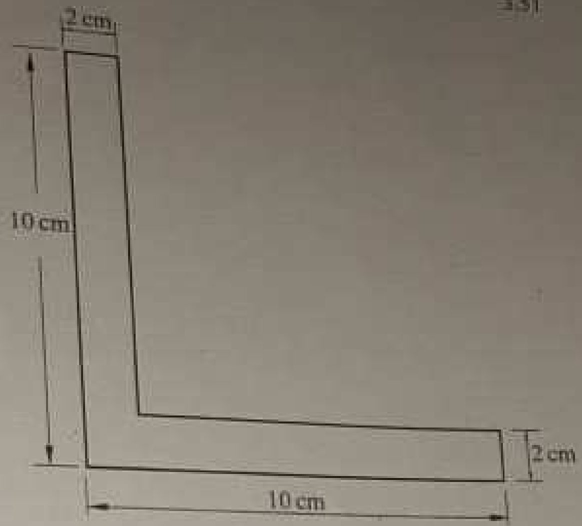

Fig. 3.85

$I_x = (I_{G_1XX} + A_1 y_1^2) + (I_{G_2XX} + A_2 y_2^2)$

$= \frac{1}{12} \times 10 \times 2^3 + 20 \times 1^2) + \frac{1}{12} \times 2 \times 8^3 + 16 \times 6^2$

$= 688\ cm^4$

$I_y = (I_{G_1YY} + A_1 x_1^2) + (I_{G_2YY} + A_2 x_2^2)$

$= (\frac{1}{12} \times 2 \times 10^3 + 20 \times 5^2) + (\frac{1}{12} \times 8 \times 2^3 + 16 \times 1^2)$

$= 688\ cm^4$

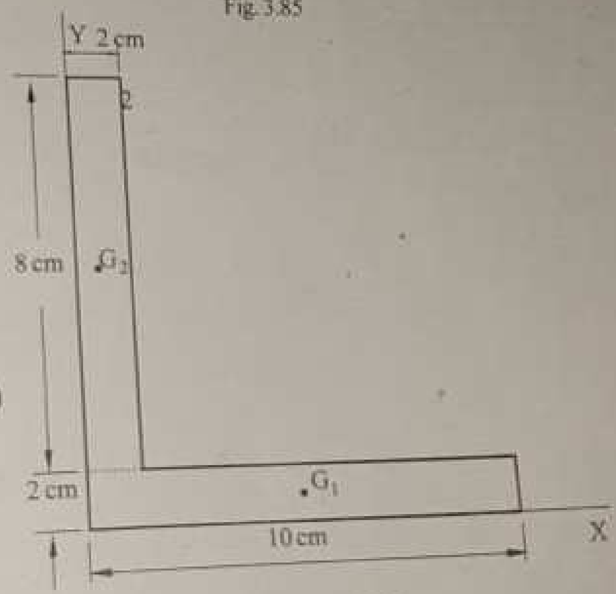

Fig. 3.86

Example 3.41

Find the moment of inertia of an unequal angle iron section of 250 mm × 20 mm about its centroidal axes.

Solution.

$A_1 = 125 \times 20 = 2500\ mm^2,\ A_2 = 230 \times 20 = 4600\ mm^2$

$x_1 = \frac{125}{2} = 62.5\ mm$

$$x_2 = \frac{20}{2} = 10 \text{ mm}$$

$$y_1 = \frac{20}{2} = 10 \text{ mm}$$

$$y_2 = 20 + \frac{230}{2} = 135 \text{ mm}$$

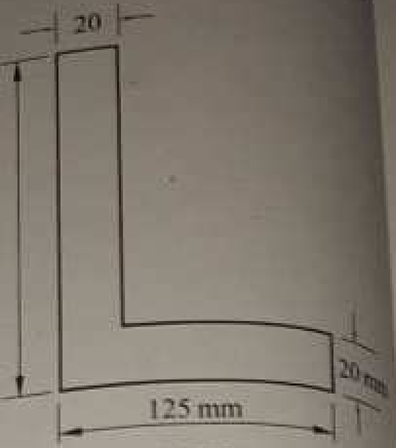

Fig 3.87

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2} = \frac{2500 \times 62.5 + 4600 \times 10}{2500 + 4600}$$

$$= 28.49 \text{ mm.}$$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{2500 \times 10 + 4600 \times 135}{2500 + 4600} = 91 \text{ mm}$$

$$IG_{1XX} = \frac{1}{12} \times 125 \times 20^3 = 83333.33 \text{ mm}^4$$

$$IG_{2XX} = \frac{1}{12} \times 120 \times 230^3 = 20278333.33 \text{ mm}^4$$

$$IG_{1YY} = \frac{1}{12} \times 20 \times 125^3 = 3255208.33 \text{ mm}^4$$

$$IG_{2YY} = \frac{1}{12} \times 230 \times 20^3 = 153333.33 \text{ mm}^4.$$

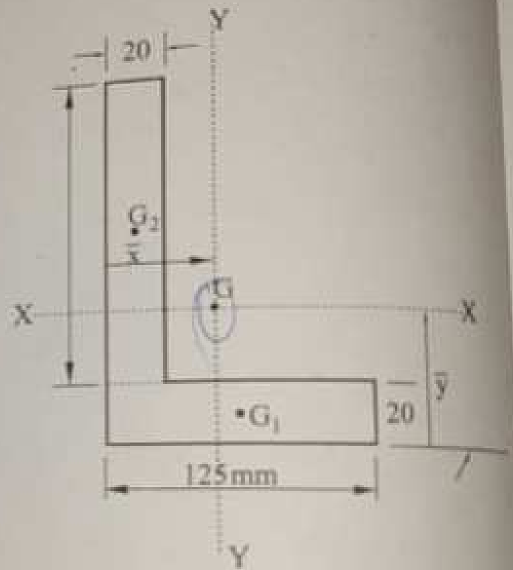

Fig. 3.88

$$IG_{XX} = I_{G1XX} + A_1 h_1^2) + (I_{G2XX} + A_2 h_2^2$$

$$h_1 = GG_1 = \bar{y} - y_1 = 91 - 10 = 81 \text{ mm}$$

$$h_2 = GG_2 = y_2 - \bar{y} = 135 - 91 = 44 \text{ mm}$$

$$IG_{XX} = (83333.33 + 2500 \times 81^2) + (20278333.33 + 4600 \times 44^2)$$

$$= 16485833.33 + 29183933.33$$

$$= 45669766.66 \text{ mm}^4$$

$$IG_{YY} = (IG_{1YY} + A_1 h_1^2) + IG_{2YY} + A_2 h_2^2$$

h_1 is the horizontal distance GG_1 and h_2 is the horizontal distance GG_2

$h_1 = x_1 - x = 62.5 - 28.49 = 34.01$ mm, $h_2 = x - x_2 = 28.49 - 10 = 18.49$ mm.

$I_{G_{YY}} = (3255208.33 + 2500 \times 34.01^2) + (153333.33 + 4600 \times 18.49^2)$

$= 7872887.37 \text{ mm}^4$

Example 3.42

Determine the moment of inertia of the T - section shown in Fig. 3.89 about the centroidal axes.

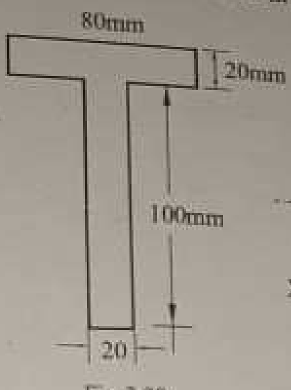

Fig. 3.89

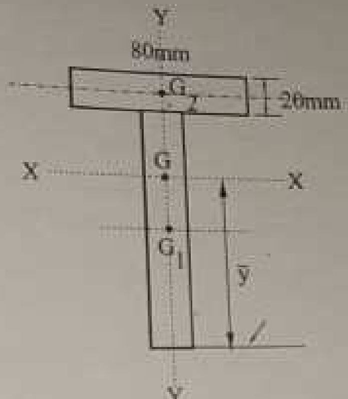

Fig. 3.90

Solution.

$A_1 = 100 \times 20 = 2000 \text{ mm}^2$

$A_2 = 80 \times 20 = 1600 \text{ mm}^2$

$y_1 = \frac{100}{2} = 50 \text{ mm}$

$y_2 = 100 + \frac{20}{2} = 110 \text{ mm}$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{2000 \times 50 + 1600 \times 110}{(2000 + 1600)}$$

$= 76.67$ mm

Moment of inertia of section (i) about its own horizontal centroidal axis is $I_{G_{1XX}}$

$$I_{G_{1XX}} = \frac{20 \times 100^3}{12} = 1666666.67 \text{ mm}.$$

Moment of inertia of section (2) about its own horizontal centroidal axis is

$$I_{G_{2XX}} = \frac{80 \times 20^3}{12} = 53333.33 \text{ mm}^4$$

$$I_{G_{XX}} = (I_{G_{1XX}} + A_1 h_1^2) + (I_{G_{2XX}} + A_2 h_2^2)$$

h_1 and h_2 are the vertical distances of G_1 and G_2 from G

$$h_1 = \bar{y} - y_1 = 76.67 - 50 = 26.67$$

$$h_2 = y_2 - \bar{y} = 110 - 76.67 = 33.33$$

$$I_{G_{XX}} = 1666666.67 + (2000 \times 26.67^2) + 53333.33 + (1600 \times 33.33^2)$$

$$= 4.92 \times 10^6 \text{ mm}^4$$

Since G_1, G_2 and G are on the same YY axis, $IG_{YY} = IG_{1YY} + IG_{2YY}$

IG_{1YY} is the M.I of section (1) about its own vertical centroidal axis, and IG_{2YY} is the M.I of section (2) about its own vertical centroidal axis.

$$IG_{YY} = \frac{100 \times 20^3}{12} + \frac{20 \times 80^3}{12}$$

$$= 9.2 \times 10^5 \text{ mm}^4$$

Example 3.43.

An I section has the following dimensions.

Top and bottom flange width 120 mm. Depth of web, 180 mm. Thickness of web and flange, 10 mm. Sketch the section and determine the area moment of inertia about centroidal XX and YY axes.

Solution

The section is symmetrical with respect to centridal X X and YY axes.

Since G_1, G_2, G_3 and G are on the same Y – Y axis, $IG_{YY} = IG_{1YY} + IG_{2YY} + IG_{3YY}$

$$= \frac{1}{12} \times 10 \times 120^3 + \frac{1}{12} \times 180 \times 10^3 + \frac{1}{12} \times 10 \times 120^3$$

$= 2895000 \text{ mm}^4$

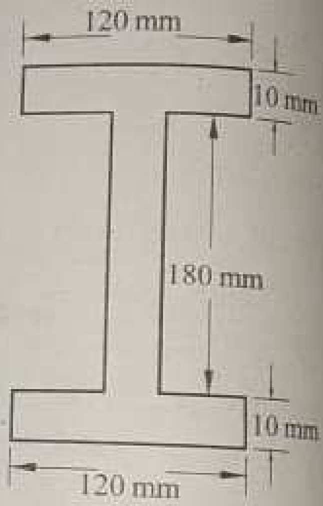

Fig.3.91

$IG_{XX} = (IG_{1XX} + A_1h_1^2) + (IG_{2XX} + A_2h_2^2) + (IG_{3}XX + A_3 h_3^2)$

h_1, h_2 and h_3 are the vertical distance GG_1, GG_2 and GG_3.

$$IG_{1XX} = IG_{3XX} = \frac{1}{12} \times 120 \times 10^3 = 10000 \text{ mm}^4$$

$$IG2_{XX} = \frac{1}{12} \times 10 \times 180^3 = 4860000 \text{ mm}^4$$

$h_1 = h_3 = 100 - 5 = 95$ mm, $h_2 = 0$

$IG_{XX} = (10000 + 1200 \times 95^2) \times 2 + (4860000 + 180 \times 10 \times 0)$

$= 26540000 \text{ mm}^4$

Alternate solution to calculate IG_{XX} (for symmetrical I section).

$IG_{XX} = IG_{1XX} - IG_{2XX} - I G_{3XX}$

$= IG_{1XX} - 2, IG_{2XX}$

$$b = \frac{B - 10}{2}$$

$$d = D - 20.$$

$$IG_{XX} = \frac{BD^3}{12} - 2 \times \frac{bd^3}{12}$$

$$= \frac{120 \times 200^3}{12} - \frac{2 \times 55 \times 180^3}{12}$$

$$= 26540000 \text{ mm}^4$$

Fig. 3.92

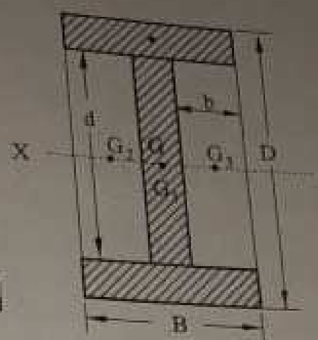

Fig. 3.93

Example 3.44

Calculate the moment of inertia of the area shown in Fig 3.94

Solution.

$a_1 = 130 \times 20 = 2600 \text{ mm}^2$

$a_2 = 300 \times 20 = 6000 \text{ mm}^2$

$a_3 = 130 \times 20 = 2600 \text{ mm}^2$

$x_1 = 85 \text{ mm}, \quad x_2 = 10 \text{ mm}, \quad x_3 = 85 \text{ mm}$

$y_1 = 140 \text{ mm}, \quad y_2 = 0, \quad y_3 = 140 \text{ mm}$

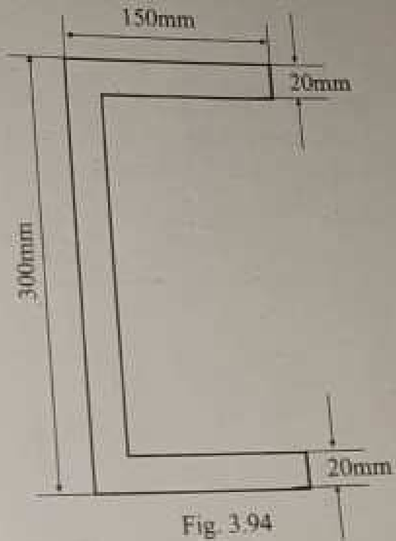

Fig. 3.94

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

$$= \frac{2600 \times 85 + 6000 \times 10 + 2600 \times 85}{2600 + 6000 + 2600}$$

$$= 44.821 \text{ mm}$$

$$IG_{YY} = (IG_{1YY} + A_1h_1^2) + (IG_{2YY} + A_2h_2^2) + (IG_{3YY} + A_3h_3^2)$$

$$= 2 \times \frac{1}{12} \times 20 \times 130^3 + 130 \times 20 \times 40.179^2 + \frac{1}{12} \times 300 \times 20^3 + 20 \times 300 \times (34.821)^2$$

$$= 23192976.19 \text{ mm}^4$$

Doubt

$$IG_{XX} = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$B = 130 + 20 = 150 \text{ mm}$

$D = 300 \text{ mm}$

$b = 130$ mm

$d = (300 - 40) = 260$ mm

$$I_{G_{XX}} = \frac{1}{12} \times 150 \times 300^3 - \frac{1}{12} \times 130 \times 260^3$$

$$= 147093333.33 \text{ mm}^4$$

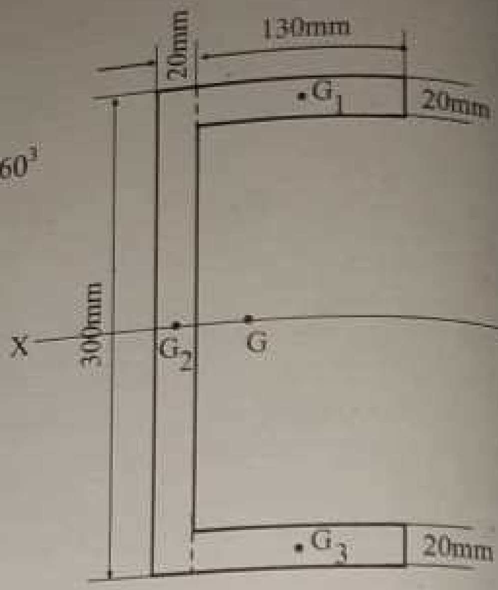

Fig. 3.95

Example 3.45

Find the moment of inertia of the cross-section of an iron beam as shown in Fig 3.96 with respect to the centroidal axes.

Solution.

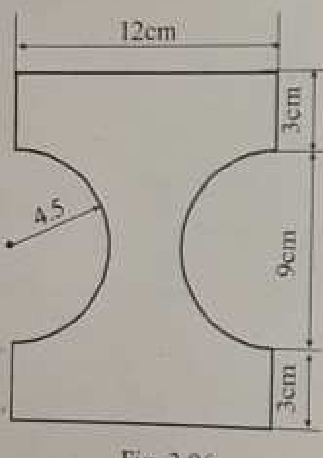

Fig. 3.96

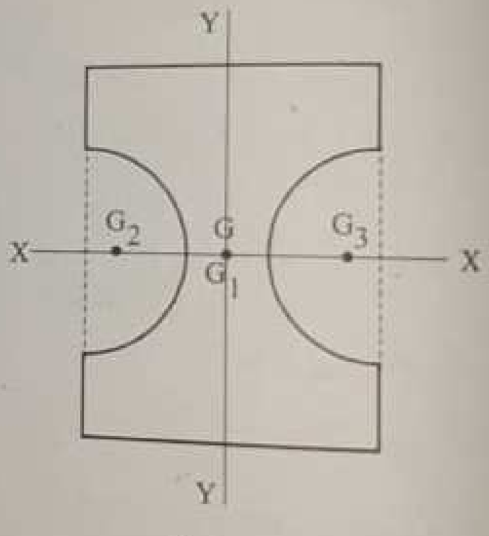

Fig. 3.97

Since G_1, G_2, G_3 and G are on the same horizontal XX axis,

$$I_{G_{XX}} = I_{G_{1XX}} - I_{G_{2XX}} - I_{G_{3XX}}$$

$$= I_{G_{1XX}} - 2 \times I_{G_{2XX}}$$

$$= \frac{1}{12} \times 12 \times 15^3 - 2 \times \frac{\pi d^4}{128}$$

$$= \frac{1}{12} \times 12 \times 15^3 - \frac{2 \times \pi \times 9^4}{128}$$

$$= 3052.94 \text{ cm}^4$$

$$IG_{YY} = IG_{1YY} - (IG_{2\,YY} + A_2\, h_2^2) - (IG_{3\,YY} + A_3\, h_3^2)$$

$$= IG_{1YY} - 2\,(IG_{2\,YY} + A_2\, h_2^2)$$

$$= \frac{1}{12} \times 15 \times 12^3 - 2\left[0.11r4 + \frac{\pi \times 4.5^2}{2} \times \left(6 - \frac{4r}{3\pi}\right)^2\right]$$

$$= \frac{1}{12} \times 15 \times 12^3 - 2\left[0.11 \times 4.5^4 + \frac{\pi \times 4.5^2}{2} \times \left(6 - \frac{4 \times 4.5}{3\pi}\right)^2\right]$$

$$= 1005.52 \text{ cm}^4$$

Example 3.46

Calculate the moment of inertia of the shaded area, as shown in Fig. 3.98, with respect to the centroidal axes.

Solution

$$a_1 = 6 \times 10^3 \times 6 \times 10^3 = 36 \times 10^6 \text{ mm}^2$$

$$a_2 = \frac{1}{2} \times 3 \times 10^3 \times 3 \times 10^3 = 4.5 \times 10^6 \text{ mm}^2$$

$$x_1 = 3 \times 10^3 \text{ mm}, \quad x_2 = \frac{2}{3} \times 3000 + 1000 = 3 \times 10^3 \text{ mm}$$

$$y_1 = 3 \times 10^3 \text{ mm}, \quad y_2 = \frac{1}{3} \times 3000 + 2000$$

$$= 3 \times 10^3 \text{ mm}$$

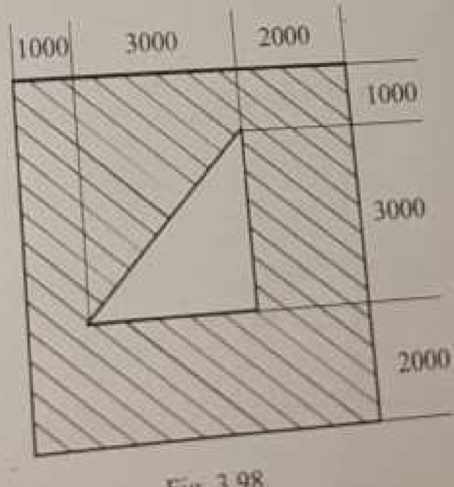

Fig. 3.98

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2} = 3 \times 10^3 \text{ mm}$$

Since G_1, G_2 and G are on the same XX axis,

$$= \frac{36 \times 10^6 \times 3 \times 10^3 - 4.5 \times 10^6 \times 3 \times 10^3}{36 \times 10^6 - 4.5 \times 10^6}$$

$IG_{XX} = IG_{1XX} - IG_{2XX}$

$$= \frac{1}{12} \times 6 \times 10^3 \times (6 \times 10^3)^3 - \frac{3 \times 10^3 \times (3 \times 10^3)^3}{36}$$

$$= 1.08 \times 10^{14} - 2.25 \times 10^{12}$$

$$= 105.75 \times 10^{12} \text{ mm}^4$$

$IG_{YY} = IG_{1YY} - IG_{2YY}$

$$= \frac{1}{12} \times 6 \times 10^3 \times (6 \times 10^3) - \frac{3 \times 10^3 \times (3 \times 10^3)^3}{36}$$

$$= 105.75 \times 10^{12}$$

$IG_{YY} = 1.0575 \times 10^{14} \text{ mm}^4$

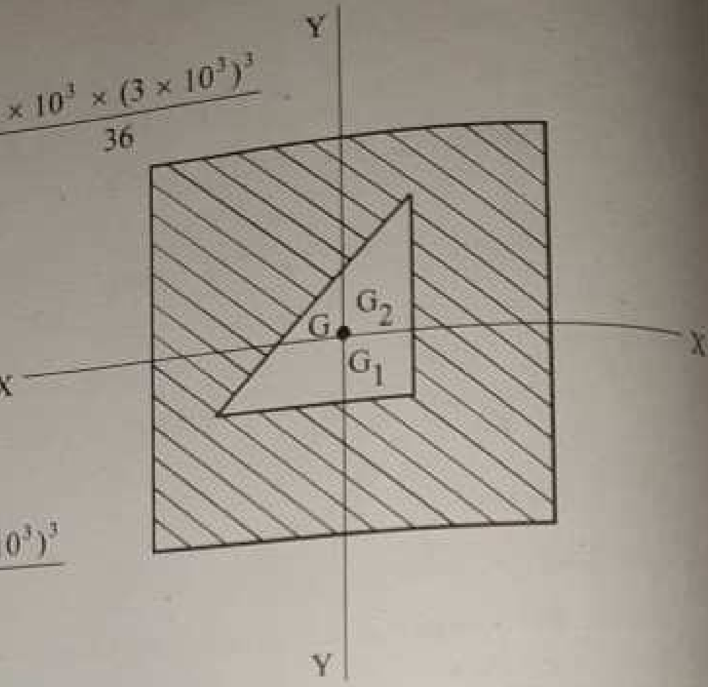

Fig. 3.99

Example 3.47

Calculate the moment of inertia of the shaded area, as shown in Fig. 3.100, with respect to the centroidal axes.

Solution.

$$a_1 = 2 \times \frac{1}{2} \times 5 \times h$$

$$\tan 60 = \frac{h}{5} \therefore h = 5 \tan 60$$

$$h = 8.66 \text{ cm}$$

$$a_1 = 2 \times \frac{1}{2} \times 5 \times 8.66 = 43.3$$

$$a_2 = 2 \times 4 = 8 \text{ cm}^2$$

$$x_1 = 0, \quad x_2 = 0$$

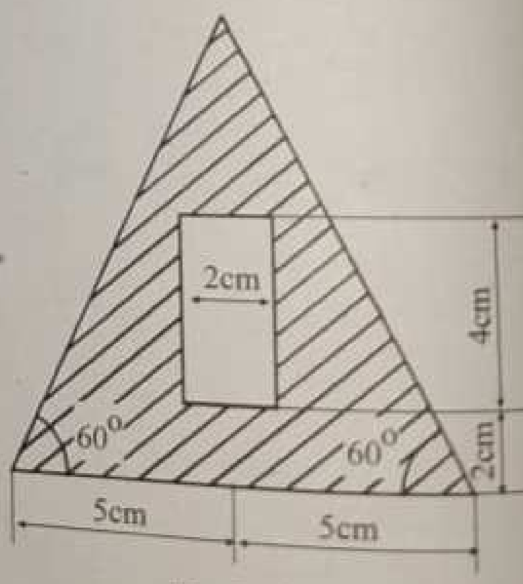

Fig. 3.100

$$y_1 = \frac{1}{3} \times h = \frac{1}{3} \times 8.66 = 2.887, \; y_2 = 2 + 2 = 4 \text{ cm}$$

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2} = 0$$

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2}$$

$$= \frac{43.33 \times 2.886 - 8 \times 4}{43.3 - 8}$$

$$= 2.635$$

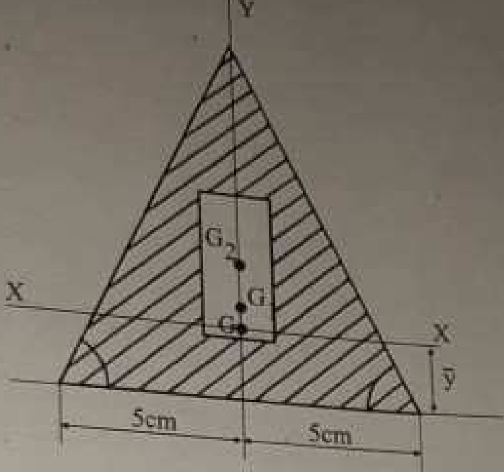

Fig. 3.101

$$IG_{XX} = (IG_{1XX} + A_1 h_1^2) - (IG_{2XX} + A_2 h_2^2)$$

$$h_1 = 2.887 - 2.635 = 0.252$$

$$h_2 = 4 - 2.635 = 1.365$$

$$= \frac{(10 \times (8.66)^3}{36} + 2 \times \frac{1}{2} \times 5 \times 8.66 \times (0.252)^2$$

$$-(\frac{1}{12} \times 2 \times 4^3 + 2 \times 4 \times (1.365)^2)$$

$$= (180.406 + 2.749) - (10.667 + 16.906)$$

$$= 157.582 \text{ cm}^4$$

$$IG_{YY} = (IG_{1YY} + A_1 h_1^2) - (IG_{2YY} + A_2 h_2^2)$$

Since $A_1 h_1^2 = A_2 h_2^2 = 0$

$$IG_{YY} = IG_{1YY} - IG_{2YY}$$

$$= \frac{8.66 \times 10^3}{36} - \frac{1}{12} \times 4 \times 2^3$$

$$IG_{XX} = 157.582 \text{ cm}^4 \quad = 240.556 - 2.667$$

$$IG_{YY} = 237.889 \text{ cm}^4 \quad = 237.889 \text{ cm}^4$$

Example 3.48

Calculate the moment of inertia of the shaded area, as shown in Fig. 3.102, with respect to the centroidal axes.

Solution.

$$a_1 = \frac{1}{2} \times 60 \times 80 = 2400 \text{ cm}^2$$

$a_2 = \pi r^2 = 176.715 \text{ cm}^2$

$x_1 = \frac{2}{3} \times 60 = 40 \text{ cm}$

$x_2 = 30 \text{ cm}$

$y_1 = \frac{2}{3} \times 80 = 53.333 \text{ cm}$

$y_2 = 50 \text{ cm}$

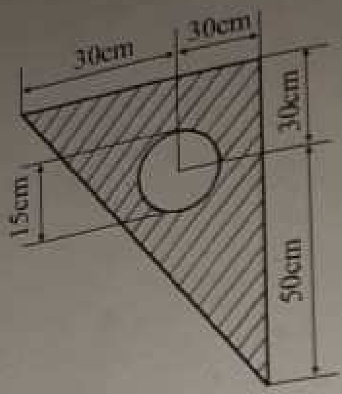

Fig. 3.102

$$\bar{x} = \frac{a_1x_1 - a_2x_2}{a_1 - a_2}$$

$$= \frac{2400 \times 53.33 - 176.715 \times 50}{2400 - 176.715}$$

$\bar{y} = 53.598 \text{ cm}$

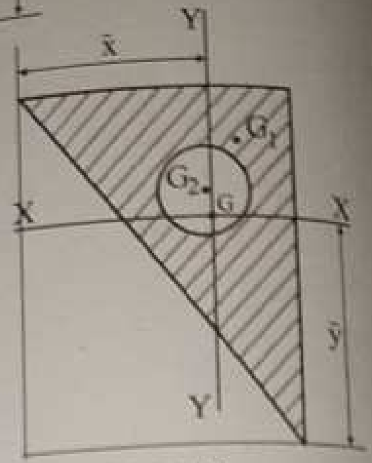

Fig. 3.103

$$IG_{XX} = (IG_{1XX} + A_1h_1^2) - (IG_{2XX} + A_2 h_2^2)$$

$$= (\frac{60 \times 80^3}{36} + \frac{1}{2} \times 60 \times 80 \times (0.265)^2) - (\frac{\pi \times 15^4}{64} + \pi \times \left[\frac{15}{2}\right]^2 \times (3.598)^2)$$

$$= (853333.333 + 168.54) - (2485.049 + 2287.677)$$

$$= 853304.5 \text{ cm}^4$$

$$IG_{YY} = (IG_{1YY} + A_1h_1^2) - (IG_{2YY} + A_2h_2^2)$$

$$= (\frac{80 \times 60^3}{36} + \frac{1}{2} \times 60 \times 80 \times (0.795)^2) - (\frac{\pi \times 15^4}{64} + \pi \times \left[\frac{15}{2}\right]^2 \times (10.795)^2)$$

$$= 480000 + 1516.86 - 2485.049 - 20592.909$$

$$IG_{YY} = 458438.902 \text{ cm}^4$$

Example 3.49

Calculate the moment of inertia of the shaded area, as shown in Fig. 3.104, with respect to the centroidal axes.

Solution.

$$a_1 = \frac{1}{2} \times 60 \times 30 = 900 \text{ cm}^2$$

$$a_2 = \frac{1}{2} \times 30 \times 15 = 225 \text{ cm}^2$$

$$x_1 = \frac{1}{3} \times 60 = 20, \quad x_2 = \frac{1}{3} \times 30 + 10 = 20$$

$$y_1 = \frac{1}{3} \times 30 = 10$$

$$y_2 = \frac{1}{3} \times 15 + 5 = 10$$

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2}$$

$$= \frac{900 \times 20 - 225 \times 20}{900 - 225} = 20 \text{ cm}$$

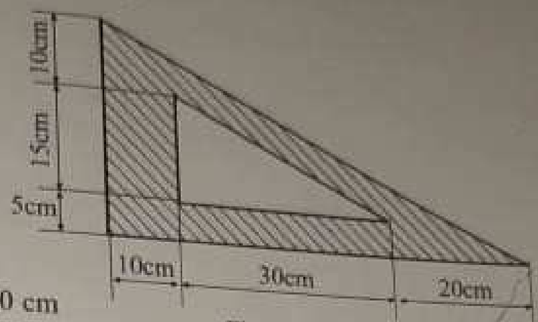

Fig. 3.104

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2} = \frac{900 \times 10 - 225 \times 10}{900 - 225} = 10 \text{ cm}$$

Since G_1, G_2 and G are on the same XX axis,

$$I_{G_{XX}} = I_{G_{1XX}} - I_{G_{2XX}}$$

$$= \frac{60 \times 30^3}{36} - \frac{30 \times 15^3}{36}$$

$$= 42187.5 \text{ cm}^4$$

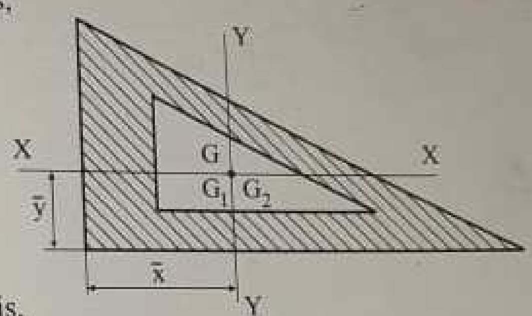

Fig. 3.105

Since G_1, G_2 and G are on the same YY axis,

$$I_{G_{YY}} = I_{G_{1YY}} - I_{G_{2YY}}$$

$$= \frac{30 \times 60^3}{36} - \frac{15 \times 30^3}{36} = 168750 \text{ cm}^4$$

Example 3.50

Find the moment of inertia of the shaded area shown in Fig 3.106 about X and Y axes.

Solution

Consider an elemental strip of thickness dx at a distance x from the Y axis. Moment of inertia of this elemental strip about its base (about OX) is $\frac{dx \times y^3}{3}$ [M.I. of a rectangle of width b and height d, about its base is $\frac{bd^3}{3}$]

$$\therefore\ Ix = \int_0^a \frac{y^3}{3}dx = \frac{1}{3}\int_0^a k^3x^6 dx = \frac{k^3}{3}\left[\frac{x^7}{7}\right]_0^a = \frac{k^3}{21}a^7$$

at A, y = b and x = a

$$y = kx^2$$

$$b = ka^2$$

$$k = \frac{b}{a^2}$$

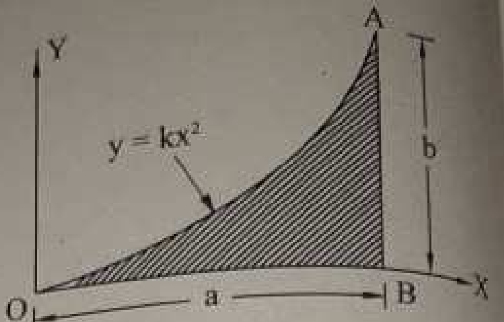

Fig. 3.106

$$\therefore Ix = \left(\frac{b}{a^2}\right)^3 \times \frac{1}{21} \times a^7 = \frac{b^3}{a^6} \times \frac{1}{21} \times a^7 = \frac{ab^3}{21}$$

$$Iy = \int ax^2 = \int_0^a ydx \times x^2 = \int_0^a kx^2dx\,x^2$$

$$= k\int_0^a x^4\,dx = k\left[\frac{x^5}{5}\right]_0^a$$

$$= k\frac{a^5}{5}$$

$$Iy = \frac{b}{a^2} \times \frac{a^5}{5}$$

$$= \frac{ba^3}{5}$$

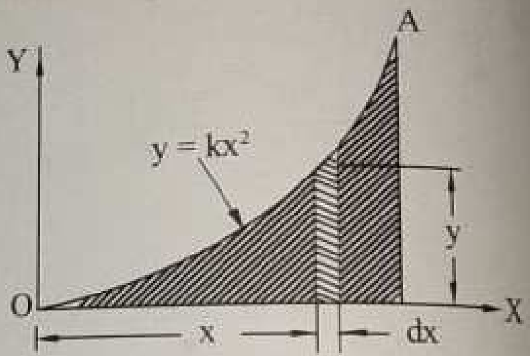

Fig. 3.107

3.10. Forces in space

When all the forces acting on a body lie in a single plane, the forces are called coplanar forces. Coplanar forces may be concurrent, non-concurrent, or parallel. When the forces acting on a body lie in different planes or when forces acting at a point in space are in different planes, the force system is said to be forces in space. Forces in space may also be parallel, concurrent or non-concurrent

Consider a force F acting at the origin O of the system of rectangular co-ordinates X, Y and Z as shown in Fig 3.108. The angles θ_x, θ_y and θ_z that the force F makes with the X, Y and Z axes define the direction of the force F. The components of the force F along the X, Y and Z directions are given by,

$$F_x = F \cos \theta_x$$

$$F_y = F \cos \theta_y \text{ and}$$

$$F_z = F \cos \theta_z$$

From Fig. 3.108, $(OE)^2 = (OD)^2 + (DE)^2$

$$= (OA)^2 + (AD)^2 + (DE)^2$$

$$= (OA)^2 + (OC)^2 + (OB)^2$$

$$F^2 = F_x^2 + F_z^2 + F_y^2$$

Magnitude of force, $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$

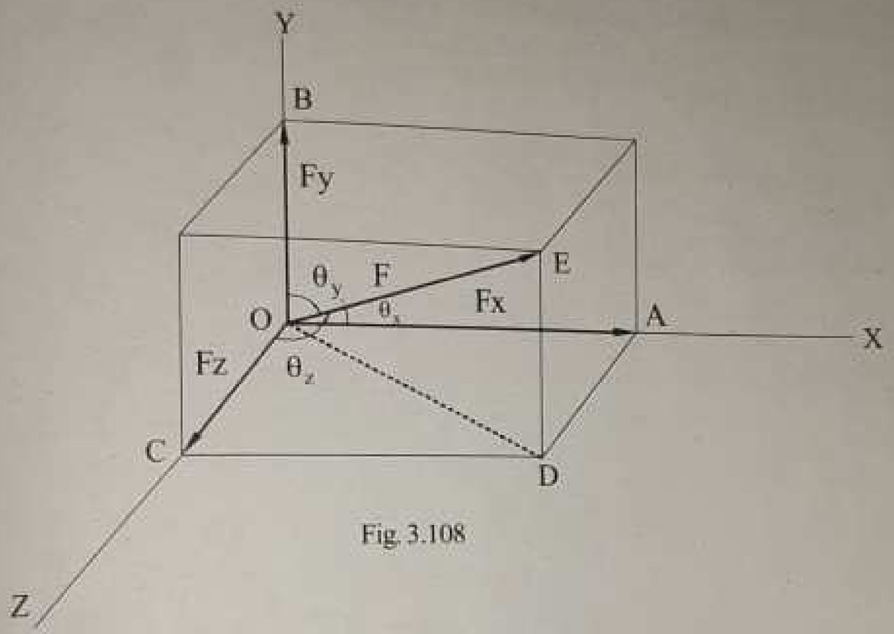

Fig. 3.108

The direction of force F acting at the origin of the system of co-ordinates can be specified by specifying a second point A on the line of action of force F which is at a distance r from the origin as shown in Fig. 3.109. The co-ordinates of A are x, y and z.

$$OA^2 = OC^2 + CA^2$$

$$= OB^2 + BC^2 + CA^2$$

$$= x^2 + z^2 + y^2$$

$$r^2 = x^2 + y^2 + z^2$$

$$\therefore r = \sqrt{x^2 + y^2 + z^2}$$

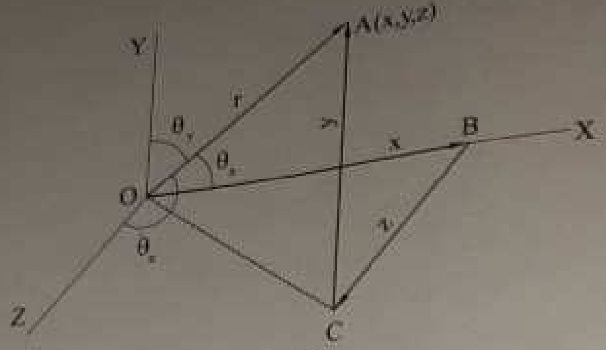

Fig. 3.109

When A and B are two points on the line of action of a force F, neither of which is at the origin, then the components x, y and z are equal to the difference between the co-ordinates of B and A. (Fig. 3.110).

$x = x_B - x_A \quad y = y_B - y_A$ and $\quad z = z_B - z_A$

The distance between A and B,

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

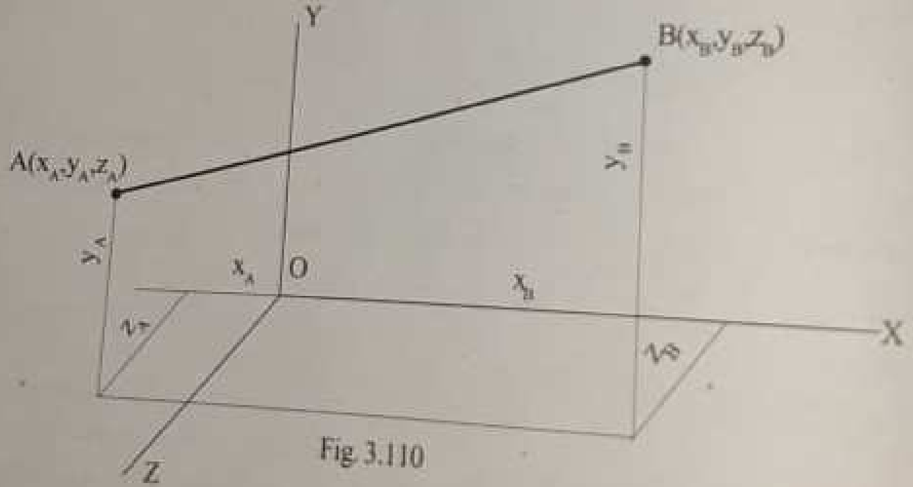

Fig 3.110

3.9. Vectorial representation of forces

The force vector $\vec{F}$, of a force of magnitude F is obtained by multiplying the magnitude of the force by a unit vector in the direction of the force. Thus force vector $\vec{F}$ = unit vector in the direction of force F × magnitude of force F

If i, j and k are the unit vectors along OX, OY and OZ directions and Fx, Fy and Fz are the components of a force along OX, OY and OZ directions then, $\vec{F} = F_x i + F_y j + F_z k$. The magnitude of this force vector, $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$. Unit vector of any vector is obtained by dividing the given vector by the magnitude of the same vector. Thus unit vector of $\vec{F}$ is $\frac{F_x i + F_y j + F_z k}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$. If A is a point in space with coordinates x, y and z, the position vector of point A with respect to the origin O, $\vec{r} = xi + yj + zk$ and the unit vector in the direction of OA is $\frac{\vec{r}}{r} = \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$. If A (x_A, y_A, z_A) and B (x_B, y_B, z_B) are two points in space then, the position vector of point B with respect to A is $(x_B - x_A)i + (y_B - y_A)j + (z_B - z_A)k$

Example 3.51 (KTU May 2017)

A force acts at the origin of a co-ordinate system in a direction defined by the angle $\theta_x = 69.3°$, $\theta_z = 57.9°$. Knowing that the Y component of the force is -174N, determine (i) the angle θ_y and (ii) the other components and the magnitude of the force.

Solution, Given $\theta_x = 69.3°$, $\theta_z = 57.9°$, $F_y = -174N$

To calculate θ_y, F_x, F_z and F

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$\cos^2 69.3 + \cos^2\theta_y + \cos^2 57.9 = 1$$

$$\cos^2\theta_y = 0.5931$$

$$\cos\theta_y = 0.77$$

$$\theta_y = 39.65 \text{ or } 140.35$$

$F_y = F\cos\theta_y = -174$. F is the magnitude of the force which is a positive value. Therefore $\cos\theta_y$ should be negative. Hence $\theta_y = 140.35°$.

$$\text{Magnitude of force } F = \frac{-174}{\cos 140.35} = 225.98N$$

$$F_x = F\cos\theta_x = 225.98 \times \cos 69.3 = 79.88N$$

$$F_z = F\cos\theta_z = 225.98 \times \cos 57.9 = 120N$$

3.10. Moment and couple

In two dimensional analysis it is easy to calculate moment of a force. It is the product of magnitude of the force and the perpendicular distance of the line of action of the force from the moment centre) In three dimensional analysis the calculation of perpendicular distance between a point and a line is tedious process. Here vector approach with cross product multiplication is convinient to calculate the moment of a force. Refer Fig. 3.111. A is a point in space and F is a force vector passing through A. Moment of this force about O,

$$Mo = F \times d$$
$$= F\, r \sin\theta$$
$$= \vec{r} \times \vec{F}$$

$$\vec{r} \times \vec{F} = (x\,i + y\,j + z\,k) \times (Fx\,i + Fy\,j + Fz\,k)$$
$$= (y\,Fz - z\,Fy)\,i + (z\,Fx - x\,Fz)\,j + (x\,Fy - y\,Fx)\,k$$
$$= M_x\,i + M_y\,j + M_z\,k$$

where $Mx = y\,Fz - z\,Fy$

$My = z\,Fx - x\,Fz$

$Mz = x\,Fy - y\,Fx$

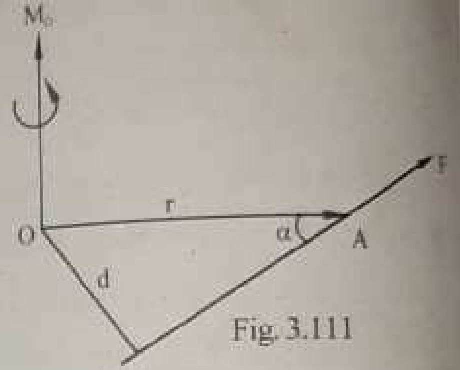

Fig. 3.111

$$\vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ x & y & z \\ Fx & Fy & Fz \end{vmatrix}$$
$$= (y\,Fz - z\,Fy)\,i + (z\,Fx - x\,Fz)\,j + (x\,Fy - y\,Fx)\,k$$

Example 3.52

A force $\vec{F} = 2\,i + 3\,j - 4\,k$ is applied at the point B (1,–1,2). Find the moment of the force about a point A (2,–1,3).

Solution.

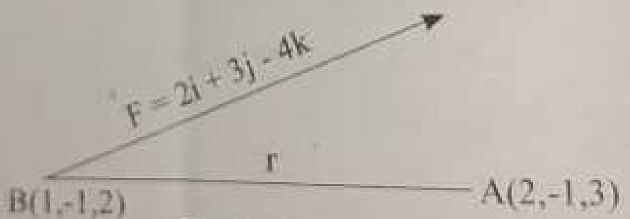

Fig. 3.112

$$x_A = 2,\ y_A = -1,\ z_A = 3$$
$$x_B = 1,\ y_B = -1,\ z_B = 2$$
$$F = 2\,i + 3\,j - 4\,k.$$

The position vector of point B with respect to A,

$$r = (x_B - x_A)\,i + (y_B - y_A)\,j + (z_B - z_A)\,k$$
$$= (1 - 2)\,i + [-1 - (-1)]\,j + (2 - 3)k$$

$$= -1i + 0j - 1k.$$

Moment of force at B about A,

$$M = r \times F$$

$$= (-1i + 0j - 1k) \times (2i + 3j - 4k).$$

$$= \begin{vmatrix} i & j & k \\ -1 & 0 & -1 \\ 2 & 3 & -4 \end{vmatrix}$$

$$= i(0+3) + j(-2-4) + k(-3-0)$$

$$M = 3i - 6j - 3k.$$

Example 3.53

A force $F = 2i + 4j - 3k$ is applied at a point A (1,1, –2). Find the moment of the force F about the point B (2, – 1, 2).

Solution.

$$x_A = 1, y_A = 1, z_A = -2$$

$$x_B = 2, y_B = -1, z_B = 2$$

$$r = (x_A - x_B)i + (y_A - y_B)j + (z_A - z_B)k$$

$$= (1-2)i + (-1-(-1))j + (-2-2)k$$

$$= -1i + 2j - 4k.$$

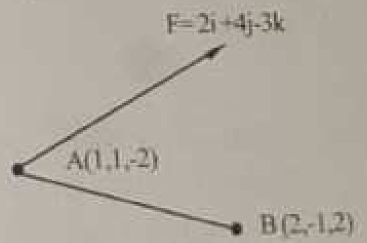

Fig. 3.113

Moment $M = r \times F$

$$= \begin{vmatrix} i & j & k \\ -1 & 2 & -4 \\ 2 & 4 & -3 \end{vmatrix}$$

$$= i[2\times(-3) - 4\times(-4)] + j\ [(-4)\times 2 - (-3)\times(-1)] + k[(-1)\times 4 - 2\times 2]$$

$$= 10i - 11j - 8k$$

Example 3.54

A force of magnitude 44 N acts through the point A (4, –1, 7) in the direction of vector $9i + 6j - 2k$. Find the moment of the force about the point B (1, –3, 2).

Solution.

Unit vector in the direction of vector $9i + 6j - 2k$ is.,

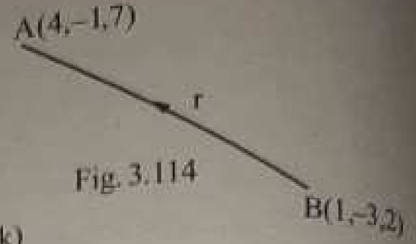

Fig. 3.114

$$\frac{9i + 6j - 2k}{\sqrt{9^2 + 6^2 + (-2)^2}} = \frac{9i + 6j - 2k}{11}$$

$$\text{The force } F = 44 \times \frac{1}{11}(9i + 6j - 2k)$$

$$F = 36i + 24j - 8k.$$

The position vector r of the point A with respect to the point B.

$$r = (4-1)i + (-1-(-3))j + (7-2)k.$$

$$= 3i + 2j + 5k.$$

$$M = r \times F = \begin{vmatrix} i & j & k \\ 3 & 2 & 5 \\ 36 & 24 & -8 \end{vmatrix}$$

$$= [(2 \times -8) - (5 \times 24)]i + [5 \times 36 - 3 \times (-8)]j + [3 \times 24 - 2 \times 36]k$$

$$= -136i + 204j + 0k.$$

$$M = -136i + 204j$$

Example 3.55

Find the moment about C (-2,3,5) of the force $F = 4i + 4j - 1k$ passing through the points A (1,-2,4) and B (5,2,3).

Solution.

Moment of force about C = r × F, where r is the position vector of any point on the force vector F with respect to the point O. It can be proved that $r_1 \times F = r_2 \times F$.

$$F = 4i + 4j - 1k.$$

$$r_1 = [1-(-2)]i + (-2-3)j + (4-5)k$$

$$= 3i - 5j - 1k.$$

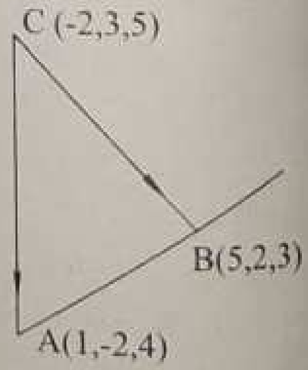

Fig. 3.115

$$r_1 \times F = \begin{vmatrix} i & j & k \\ 3 & -2 & -1 \\ 4 & 4 & -1 \end{vmatrix}$$

$$= (5+4)i + (-4+3)j + (12+20)k$$

$$= 9i - j + 32k.$$

$$r_2 = [5-(-2)]\,i + (2-3)\,j + (3-5)\,k$$
$$= 7i - j - 2k$$

$$M = r_2 \times F = \begin{vmatrix} i & j & k \\ 7 & -1 & -2 \\ 4 & 4 & -1 \end{vmatrix}$$

$$= (1+8)\,i + (-8+7)\,j + (28+4)\,k = 9i - j + 32k$$

Example 3.56

A force of 60 kN passes from point A(0,2,3) to point B(7,0,5). Find the moment of this force about a point C(7,4,3).

Solution

$$x_A = 0,\ y_A = 2,\ z_A = 3$$
$$x_B = 7,\ y_B = 0,\ z_B = 5$$
$$x_C = 7,\ y_C = 4,\ z_C = 3$$

Position vector of line AB $= (7-0)\,i + (0-2)\,j + (5-3)\,k$
$= 7i - 2j + 2k$

Unit vector along AB $= \dfrac{7i - 2j + 2k}{\sqrt{7^2 + 2^2 + 2^2}} = 0.93i - 0.26j + 0.26k$

Force vector along AB, $\vec{F} = 60(0.93i - 0.26j + 0.26k)$

$= 55.8\,i - 15.6\,j + 15.6\,k$

Position vector of CB $= (7-7)\,i + (0-4)\,j + (5-3)\,k$
$= 0\,i - 4j + 2k$

Moment of force about C, $\vec{M}_c = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ 0 & -4 & 2 \\ 55.8 & -15.6 & 15.6 \end{vmatrix}$

$= (-4\times 15.6 + 2\times 15.6)\,i - (2\times 55.8)\,j + (0 + 223.2)\,k$

$= -31.2\,i + 111.6\,j + 223.2\,k$

Magnitude of moment $M = \sqrt{31.22^2 + 111.6^2 + 223.2^2}$

$= 251.49$ kNm

Couple

Two parallel forces, equal in magnitude and opposite in direction constitute a couple. Since the two forces are equal and opposite the sum of forces $\Sigma F = 0$. Magnitude of moment of a couple is a constant and is equal to the product of magnitude of the force and distance between the two forces. Moment of a couple, $M = F \times d$.

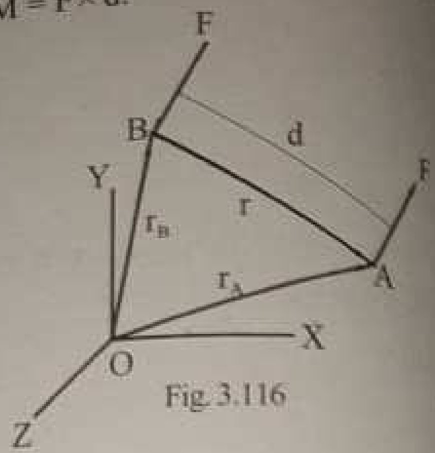

Fig 3.116

Moment of force F at A about $O = r_A \times F$
Moment of force F at B about $O = r_B \times (-F)$
Sum of moments about O,

$$M_O = \vec{r}_A \times \vec{F} - \vec{r}_B \times \vec{F}$$
$$= (\vec{r}_A - \vec{r}_B) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$

From the vector diagram OBA

$\vec{r}_B + \vec{r} = \vec{r}_A$, $\vec{r}_A - \vec{r}_B = \vec{r}$

Example 3.57

Two forces $\vec{F}_1 = 50\,i + 80\,j + 100\,k$ and $\vec{F}_2 = -50\,i - 80\,j - 100\,k$ acts at point A (0.7, 1.5, 1) and B (1, 0.9, -1) respectively. Calculate the moment of the forces and perpendicular distance between the forces.

Solution

Given,

$$\vec{F}_1 = 50\,i + 80\,j + 100\,k$$
$$\vec{F}_2 = -50\,i - 80\,j - 100\,k$$
$$x_A = 0.7,\ y_A = 1.5,\ z_A = 1$$
$$x_B = 1,\ y_B = 0.9,\ z_B = -1$$

Moment of couple, $\vec{M} = \vec{r} \times \vec{F}$

$$\vec{r} = (x_A - x_B)\,i + (y_A - y_B)\,j + (z_A - z_B)\,k$$
$$= -0.3\,i + 0.6\,j + 2\,k$$

$$\vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ -0.3 & 0.6 & 2 \\ 50 & 80 & 100 \end{vmatrix}$$

$$= (60 - 160)\,i + (100 + 30)\,j + (-24 - 30)\,k$$
$$\vec{M} = 100\,i + 130\,j - 54\,k$$

Magnitude of $\vec{M} = \sqrt{100^2 + 130^2 + 54^2}$
$$= 172.67$$

Magnitude of force $F = \sqrt{50^2 + 80^2 + 100^2}$
$$= 137.48$$

$$\text{Moment of Couple} = F \times d$$

$$\therefore d = \frac{M}{F}$$

$$= \frac{172.67}{137.48}$$

$$= 1.26\text{m}$$

3.11. Resultant of concurrent forces in space.

Resultant of concurrent force in space can be obtained by summing their rectangular components. Let $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ be concurrent forces in space.

$$\vec{F}_1 = F_{1x}i + F_{1y}j + F_{1z}k$$

$$\vec{F}_2 = F_{2x}i + F_{2y}j + F_{2z}k$$

$$\vec{F}_3 = F_{3x}i + F_{3y}j + F_{3z}k$$

Resultant of these forces,

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$R = R_x i + R_y j + R_z k$$

$$= F_{1x}i + F_{1y}j + F_{1z}k + F_{2x}i + F_{2y}j + F_{2z}k + F_{3x}i + F_{3y}j + F_{3z}k$$

$$= (F_{1x} + F_{2x} + F_{3x})i + (F_{1y} + F_{2y} + F_{3y})j + (F_{1z} + F_{2z} + F_{3z})k$$

$$= \Sigma F_x i + \Sigma F_y j + \Sigma F_z k$$

$$R_x = \Sigma F_x$$

$$R_y = \Sigma F_y$$

$$R_z = \Sigma F_z$$

The magnitude of the resultant $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$ and θ_x, θ_y and θ_z are given by cos $\theta_x = \frac{R_x}{R}$; $\cos\theta_y = \frac{R_y}{R}$; $\cos\theta_z = \frac{R_z}{R}$.

Example 3.58

Forces 30 k N, 20 k N, 25 k N and 40 k N are concurrunt at origin and are directed through the points A (2, 1, 6), B (4, -2, 5), C (-3, -2, 1) and D (5, 1, -20). Determine the resultant of the forces.

Solution:

Position vector of point A, $\vec{r}_A = 2\vec{i} + 1\vec{j} + 6\vec{k}$

Position vector of point B, $\vec{r}_B = 4\vec{i} - 2\vec{j} + 5\vec{k}$

Position vector of point C, $\vec{r}_C = -3\vec{i} - 2\vec{j} + \vec{k}$

Position vector of point D, $\vec{r}_D = 5\vec{i} + 1\vec{j} - 2\vec{k}$

Unit vector along OA $= \dfrac{2\vec{i} + 1\vec{j} + 6\vec{k}}{\sqrt{2^2 + 1^2 + 6^2}} = 0.31\vec{i} + 0.16\vec{j} + 0.94\vec{k}$

Unit vector along OB $= \dfrac{4\vec{i} - 2\vec{j} + 5\vec{k}}{\sqrt{4^2 + 2^2 + 5^2}} = -0.6\vec{i} - .3\vec{j} + 0.75\vec{k}$

Unit vector along OC $= \dfrac{-3\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{3^2 + 2^2 + 1^2}} = -0.8\vec{i} - 0.53\vec{j} + 0.27\vec{k}$

Unit vector along OD $= \dfrac{5\vec{i} + 1\vec{j} - 2\vec{k}}{\sqrt{5^2 + 1^2 + 2^2}} = 0.91\vec{i} + 0.18\vec{j} - 0.36\vec{k}$

Force vector along OA, $F_1 = 30\,[0.31\vec{i} + 0.16\vec{j} + 0.94\vec{k}]$

$= 9.3\vec{i} + 4.8\vec{j} + 28.2\vec{k}$

Force vector along OB, $F_2 = 20\,[0.6\vec{i} - 0.3\vec{j} + 0.75\vec{k}]$

$= 12\vec{i} - 6\vec{j} + 15\vec{k}$

Force vector along OC, $F_3 = 25\,[-0.8\vec{i} - 0.53\vec{j} + 0.27\vec{k}]$

$= -20\vec{i} - 13.25\vec{j} + 6.75\vec{k}$

Force vector along OD, $F_4 = 40\,[0.91\vec{i} + 0.18\vec{j} - 0.36\vec{k}]$

$= 36.4\vec{i} + 7.2\vec{j} - 14.4\vec{k}$

Resultant force $R = \overline{F_1} + \overline{F_2} + \overline{F_3} + \overline{F_4}$

$= 9.3\vec{i} + 4.8\vec{j} + 28.2\vec{k} + 12\vec{i} - 6\vec{j} + 15\vec{k} - 20\vec{i} - 13.25\vec{j} + 6.75\vec{k} + 36.4\vec{i} + 7.2\vec{j} - 14.4\vec{k}$

ii) Expr

$$= 37.7\vec{i} + 31.25\vec{j} + 31.55\vec{k}$$

$$R = \sqrt{37.7^2 + 31.25^2 + 31.55^2}$$

$$= 60.51 kN$$

Inclination of resultant,

$$\theta_x = \cos^{-1}\left(\frac{R_x}{R}\right) = \cos^{-1}\left(\frac{37.7}{60.5}\right) = 51.46^\circ$$

$$\theta_y = \cos^{-1}\left(\frac{R_y}{R}\right) = \cos^{-1}\left(\frac{31.25}{60.5}\right) = 58.91^\circ$$

$$\theta_z = \cos^{-1}\left(\frac{R_z}{R}\right) = \cos^{-1}\left(\frac{35.55}{60.5}\right) = 54.02^\circ$$

Example 3.59

Two cables AB and AC are attached at A as shown in Fig 3.117. Determine the resultant of the forces exerted at A by the two cables, if the tension is 2000 N in the cable AB and 1500 N in the cable AC.

Solution.

The coordinates of A, B and C are,

A (52,0,0),

B (0,50,40) and C (0, 62, –50)

$$x_A = 52, y_A = 0, z_A = 0$$

$$x_B = 0, y_B = 50, z_B = 40$$

$$x_C = 0, y_C = 62, z_C = -50$$

$$r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

$$= \sqrt{(-52)^2 + 50^2 + 40^2}$$

$$= 82.5 \text{ m}$$

$$r_{AC} = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2}$$

$$= \sqrt{(-52)^2 + 62^2 + (-50)^2}$$

$$= 95.12 \text{ m.}$$

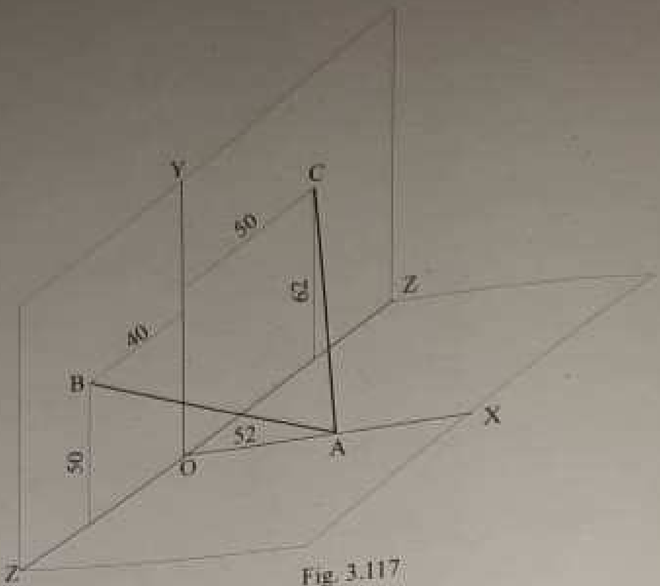

Fig. 3.117

Unit vector in the direction of AB,

$$= \frac{(0-52)i + (50-0)j + (40-0)k}{\sqrt{(-52)^2 + (50)^2 + (40)^2}}$$

$$= \frac{-52i + 50j + 40k}{82.49}$$

$$\text{Force vector along AB} = 2000\left(\frac{-52i + 50j + 40k}{82.49}\right)$$

$$= -1260.75\ i + 1212.27\ j + 969.81\ k$$

Unit vector in the direction of AC, $= \frac{(0-52)i + (62-0)j + (-50-0)k}{\sqrt{(-52)^2 + (62)^2 + (-50)^2}}$

$$= \frac{-52i + 62j - 50k}{95.12}$$

$$\text{Force vector along AC} = 1500\left(\frac{-52i + 62j - 50k}{95.12}\right)$$

$$= -820.02\ i + 977.71\ j - 788.48\ k$$

Resultant force at A, $R = F_{AB} + F_{AC}$

$$= (-1260.75\ i + 1212.27\ j + 969.81\ k) + (-820.02\ i + 977.71\ j - 788.48\ k)$$

$$= -2080.78\ i + 2189.98\ j + 181.33\ k$$

The magnitude of resultant at A, $R = \sqrt{(-2080.78)^2 + (2189.98)^2 + (181.33)^2}$

$= 3026.31\ N$

Direction of resultant is given by,

$$\cos\theta_x = \frac{R_x}{R} = \frac{-2080.78}{3026.31}$$

$$\theta_x = 133.44^\circ$$

$$\cos\theta_y = \frac{R_y}{R} = \frac{2189.98}{3026.31}$$

$$\theta_y = 43.64^\circ$$

$$\cos\theta_Z = \frac{R_Z}{R} = \frac{181.33}{3026.31}$$

$$\theta_z = 86.56^\circ$$

Example 3.60

Three forces 500N, 700N and 800N are acting along the three diagonals of adjacent faces of a cube of side 2 m as showin in Fig. 3.118. Determine the resultant of the forces.

Solution

Co-ordinate of A (2, 0, 2)

Co-ordinate of B (2, 2, 0)

Co-ordinate of C (0, 2, 2)

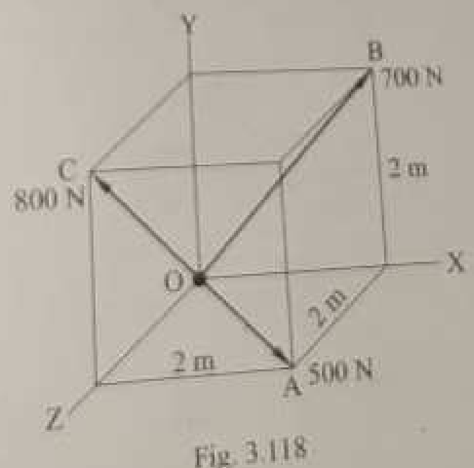

Fig. 3.118

Unit vector along OA $= \frac{2i+0j+2k}{\sqrt{2^2+0^2+2^2}} = 0.71i+0j+0.71k$

Force vector along OA $= 500\,(0.71\,i + 0\,j + 0.71\,k)$

$= 355\,i + 0\,j + 355\,k$

Unit vector along OB $= \frac{2i+2j+0k}{\sqrt{2^2+2^2+0^2}} = 0.71i+0.71j+0k$

Force vector along OB $= 700\,(0.71\,i + 0.71\,j + 0\,k)$

$= 497\,i + 497\,j + 0\,k$

Unit vector of OC $= \frac{0i+2j+2k}{\sqrt{0^2+2^2+2^2}} = 0i+0.71j+0.71k$

Force vector along OC $= 800\,(0i + 0.71\,j + 0.71\,k)$

$= 0i + 569\,j + 569\,k$

Resultant force R $= F_{OA} + F_{OB} + F_{OC}$

$= (355\,i + 0\,j + 355\,k) + (497\,i + 497\,j + 0\,k) + (0\,i + 569\,j + 569\,k)$

$R = 852\,i + 1066\,j + 924\,k$

Magnitude of resultat

$$R = \sqrt{852^2 + 1066^2 + 924^2}$$

$$= 1648\text{ N}$$

The direction cosines are given by

$$\cos\theta_x = \frac{\Sigma F_x}{\Sigma F} = \frac{852}{1648}$$

$$\theta_x = 58.87^\circ$$

$$\cos\theta_y = \frac{\Sigma F_y}{\Sigma F} = \frac{1066}{1648}$$

$$\theta_y = 49.7^\circ$$

$$\cos\theta_z = \frac{\Sigma F_z}{\Sigma F} = \frac{924}{1648}$$

$$\theta_z = 55.90^\circ$$

Example 3.61

A post is held in vertical position by three cable AB, AC and AD as shown in Fig 3.119. If the tension in the cable AB is 40 N, calculate the required tension in AC and AD so that the resultant of the three forces applied at A is vertical.

Solution.

For the resultant force to be vertical, the components R_x and R_z must be zero.

Co-ordinates of A are (0, 48, 0). Co-ordinates of B are (16, 0, 12). Co-ordinates of C are (16, 0, – 24). Co-ordinates of D are (–14, 0, 0).

$$x_A = 0,\ y_A = 48,\ z_A = 0$$

$$x_B = 16,\ y_B = 0,\ z_B = 12$$

$$x_C = 16,\ y_C = 0,\ z_C = -24$$

$$x_D = -14,\ y_D = 0,\ z_D = 0$$

$$r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

$$r_{AB} = \sqrt{(16)^2 + (-48)^2 + 12^2}$$

$$= 52\ \text{m}$$

$$r_{AC} = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2}$$

$$r_{AC} = \sqrt{16^2 + (-48)^2 + (-24)^2}$$

$$= 56\ \text{m}$$

$$r_{AD} = \sqrt{(x_D - x_A)^2 + (y_D - y_A)^2 + (z_D - z_A)^2}$$

$$r_{AD} = \sqrt{(-14)^2 + (-48)^2 + 0^2}$$

$$= 50\ \text{m}$$

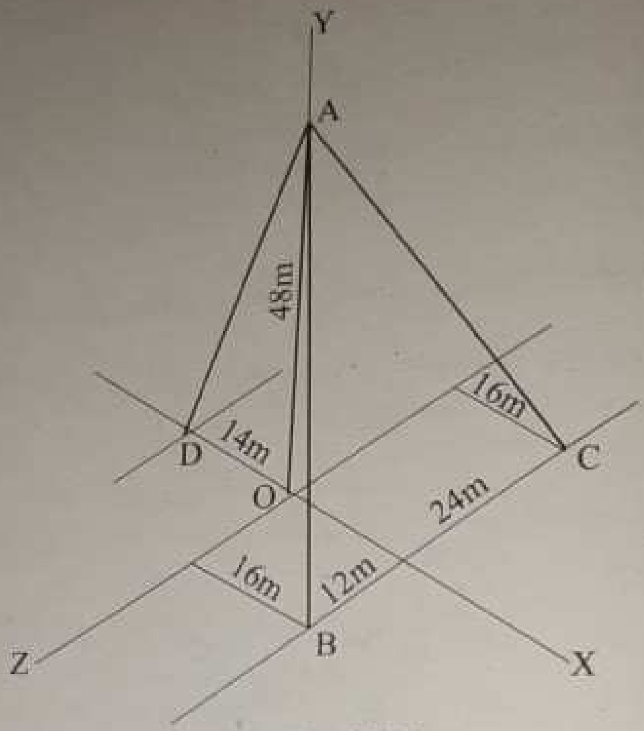

Fig. 3.119

Unit vector in the direction of AB,

$$= \frac{(16-0)i + (0-48)j + (12-0)k}{\sqrt{(16)^2 + (-48)^2 + (12)^2}}$$

$$= \frac{16i - 48j + 12k}{52}$$

Force vector along AB $= 40\left(\frac{16i - 48j + 12k}{52}\right)$

$$= 12.31\,i - 36.92\,j + 9.23\,k$$

Unit vector in the direction of AC,

$$= \frac{(16-0)i + (0-48)j + (0-24)k}{\sqrt{(16)^2 + (-48)^2 + (-24)^2}}$$

$$= \frac{16i - 48j - 24k}{56}$$

Force vector along AC $= F_{AC}\left(\frac{16i - 48j - 24k}{56}\right)$

$$= 0.29\,F_{AC}\,i - 0.86\,F_{AC}\,j - 0.43\,F_{AC}\,k$$

Unit vector in the direction of AD,

$$= \frac{(0-14)i + (0-48)j + (0-0)k}{\sqrt{(-14)^2 + (-48)^2 + (0)^2}}$$

$$= \frac{-14i - 48j + 0k}{50}$$

Force vector along AC $= F_{AD}\left(\frac{-14i - 48j + 0k}{50}\right)$

$$= -0.28\,F_{AD}\,i - 0.96\,F_{AD}\,j + 0\,k$$

Resultant force at A, $R = F_{AB} + F_{AC} + F_{AD}$

$$= (12.31\,i - 36.92\,j + 9.23\,k) + (0.29\,F_{AC}\,i - 0.86\,F_{AC}\,j - 0.43\,F_{AC}\,k) + (-0.28\,F_{AD}\,i - 0.96\,F_{AD}\,j + 0k)$$

$$= (12.31 + 0.29\,F_{AC} - 0.28\,F_{AD})\,i + (-36.96 - 0.86\,F_{AC} - 0.96\,F_{AD})\,j + (9.23 - 0.43\,F_{AC} + 0)\,k$$

For the resultant to the vertical, the X and Z components must be zero. i.e.,

$$Fz = (9.23 - 0.43\,F_{AC} + 0) = 0$$

$$0.43\,F_{AC} = 9.23$$

$$F_{AC} = \frac{9.23}{0.343}$$

$$= 21.47\text{ N}$$

$$Fx = (12.31 + 0.29\,F_{AC} - 0.28\,F_{AD}) = 0$$

$$0.28\,F_{AD} = 12.31 + 0.29\,F_{AC}$$

$$= 1231 + 0.29 \times 21.47$$

$$F_{AD} = 66.20\text{ N}$$

3.12. Equilibrium equations of concurrent forces in space.

A body will be in equilibrium if the resultant of all the forces acting on it is zero. For equilibrium the components R_x, R_y and R_z must be zero. ie.,

$$\Sigma F_x = R_x = 0,\ \Sigma F_y = R_y = 0,\ \Sigma F_z = R_z = 0.$$

Example 3.62

Three cables are joined at A, where two forces P and Q are applied in X and Z directions as shown in Fig. 3.120. If P = 3.5 kN and Q = 1.5 kN, determine the tension in each cable.

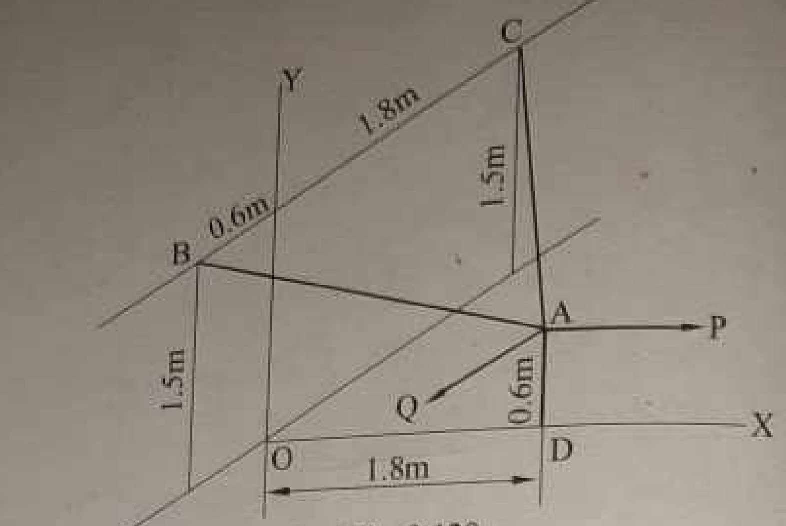

Fig. 3.120

Solution.

Co-ordinates of A are (1.8, 0.6, 0) Co-ordinates of B are (0, 1.5, 0.6) Co-ordinates of C are(0, 1.5, −1.8).Co-ordinates, of D are (1.8, 0, 0)

$$x_A = 1.8, y_A = 0.6, z_A = 0$$

$$x_B = 0, y_B = 1.5, z_B = 0.6$$

$$x_C = 0, y_C = 1.5, z_C = -1.8$$

$$x_D = 1.8, y_D = 0, z_D = 0$$

$$r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

$$r_{AB} = \sqrt{(-1.8)^2 + (0.9)^2 + (0.6)^2}$$

$$r_{AC} = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2}$$

$$= \sqrt{(0-1.8)^2 + (1.5-0.6)^2 + (-1.8-0)^2}$$

$$= \sqrt{(-1.8)^2 + (0.9)^2 + (-1.8)^2}$$

$$= 2.7 \text{ m}$$

$$r_{AD} = \sqrt{(x_D - x_A)^2 + (y_D - y_A)^2 + (z_D - z_A)^2}$$

$$= \sqrt{(1.8-1.8)^2 + (0-0.6)^2 + (0-0)^2}$$

$$= \sqrt{(0)^2 + (-0.6)^2 + (0)^2}$$

$$= 0.6 \text{ m}$$

Unit vector in the direction of AB,

$$= \frac{(0-1.8)i + (1.5-0.6)j + (0.6-0)k}{\sqrt{(-1.8)^2 + (-0.9)^2 + (0.6)^2}}$$

$$= \frac{-1.8i + 0.9j - 0.6k}{2.1}$$

$$\text{Force vector along AB} = T_{AB}\left(\frac{-1.8i + 0.9j - 0.6k}{2.1}\right)$$

$$= T_{AB}(-0.86\,i + 0.43\,j + 0.29\,k)$$

Unit vector in the direction of AC,

$$= \frac{(0-1.8)i + (1.5-0.6)j + (0-1.8)k}{\sqrt{(-1.8)^2 + (-0.9)^2 + (-1.8)^2}}$$

$$= \frac{-1.8i + 0.9j - 1.8k}{2.7}$$

$$\text{Force vector along AC} = T_{AC}\left(\frac{-1.8i + 0.9j - 1.8k}{2.7}\right)$$

$$= T_{AC}(-0.76\,i + 0.33\,j - 0.67\,k)$$

Unit vector in the direction of AD,

$$= \frac{(0-0)\,i + (0-0.6)\,j + (0-0)\,k}{\sqrt{(0)^2 + (-0.6)^2 + (0)^2}}$$

$$= \frac{0i - 0.6j + 0k}{0.6}$$

Force vector along AD $= T_{AD}\left(\frac{0\,i - 0.6\,j + 0\,k}{0.6}\right)$

$= 0\,i - 1\,T_{AD}\,j + 0\,k$

Force vector along P

$= 3.5\,i + 0\,j + 0\,k$

Force vector along Q

$= 0\,i + 0\,j + 1.5\,k$

For the equilibrium of point A, $\Sigma F_x = R_x = 0$

$-0.86\,T_{AB} - 0.67\,T_{AC} + 0 + 3.5 + 0 = 0$

$0.86\,T_{AB} + 0.67\,T_{AC} = 3.5$ ...... (i)

$\Sigma F_y = R_y = 0$

$0.43\,T_{AB} + 0.33\,T_{AC} - T_{AD} + 0 + 0 = 0$

$0.43\,T_{AB} + 0.33\,T_{AC} - T_{AD} = 0$ ..... (ii)

$\Sigma F_z = R_z = 0$

$0.29\,T_{AB} - 0.67\,T_{AC} + 0 + 1.5 = 0$

$0.29\,T_{AB} - 0.67\,T_{AC} = -1.5$ ..... (iii)

Solving eqns (i), (ii) and (iii)

$T_{AB} = 1.74$ kN

$T_{AC} = 3$ kN

$T_{AD} = 1.74$ kN

Example 3.63

A tripod supports a load of 2 kN as shown in Fig. 3.121. The ends A, B and C are in the X-Z plane. Find the force in the three legs of the tripod.

Solution:

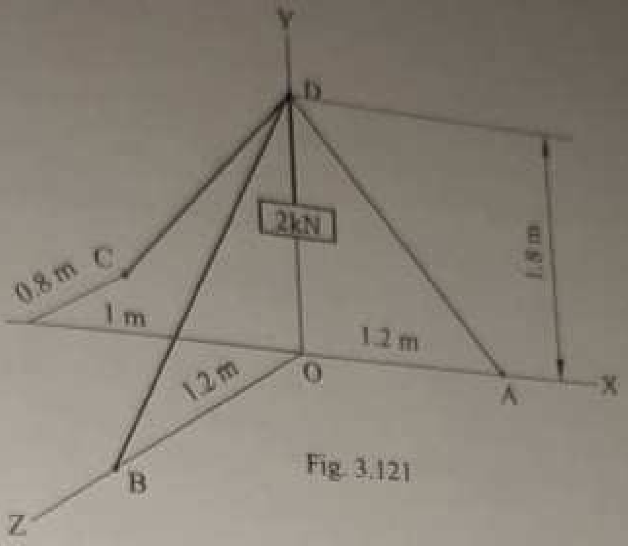

Fig. 3.121

Co-ordinates of A are (1.2, 0, 0)

Co-ordinates of B are (0, 0, 1.2)

Co-ordinates of C are (–1, 0, –0.8)

Co-ordinates of D are (0, 1.8, 0)

$$\text{Position vector AD} = (0 - 1.2)\,i + (1.8 - 0)\,k + (0 - 0)\,k$$

$$= -1.2\,i + 1.8\,k + 0\,k$$

$$\text{Unit vector along AD} = \frac{-1.2i + 1.8j + 0k}{\sqrt{(-1.2)^2 + (1.8)^2 + (0)^2}}$$

$$= -0.56\,i + 0.83\,j + 0\,k$$

$$\text{Force vector along AD} = F_{AD}\,(-0.56\,i + 0.83\,j + 0k)$$

$$\text{Position vector BD} = (0 - 0)\,i + (1.8 - 0) + (0 - 1.2)\,k$$

$$= 0\,i + 1.8\,j - 1.2\,k$$

$$\text{Unit vector along AD} = \frac{0\,i + 1.8j - 1.2k}{\sqrt{0^2 + 1.8^2 + (-1.2)^2}}$$

$$= 0\,i + 0.83\,j - 0.56\,k$$

$$\text{Force vector along BD} = F_{BD}\,(0\,i + 0.83\,j - 0.56\,k)$$

$$\text{Position vector CD} = [0 - (-1)]\,i + (1.8 - 0)\,j + [0 - (-0.8)]\,k$$

$$= i + 1.8\,j + 0.8\,k$$

Unit vector along CD $= \frac{0i + 1.8j + 0.8k}{\sqrt{1^2 + 1.8^2 + 0.5^2}}$

$= 0.45\,i + 0.81\,j + 0.36\,k$

Force vector along CD $= F_{CD}\,(0.45\,i + 0.81\,j + 0.36\,k)$

Consider the equilibrium of point D, sum of force vectors acting at D must be zero.

$F_{AD}\,(-0.56\,i + 0.83\,j + 0\,k) + F_{BD}\,(0\,i + 0.83\,j - 0.56\,k)$

$+ F_{CD}\,(0.45\,i + 0.81\,j + 0.36\,k) - 2\,j = 0$

$(-0.56\,F_{AD} + 0 + 0.45\,F_{CD} + 0)\,i + (0.83\,F_{AD} + 0.83\,F_{BD} + 0.81\,F_{CD} - 2)\,j + (0 - 0.56\,F_{BD} +$
$0.36\,F_{CD} + 0)\,k = 0$

$$-0.56\,F_{AD} + 0.45\,F_{CD} = 0$$

$$0.83\,F_{AD} + 0.83\,F_{BD} + 0.81\,F_{CD} = 2$$

$$-0.56\,F_{BD} + 0.36\,F_{CD} = 0$$

Solving the above three equations

$$F_{AD} = 802\text{ kN}$$

$$F_{BD} = 642\text{ kN and}$$

$$F_{CD} = 990\text{ kN}$$

3.13. Resultant of non - concurrent forces in space.

Consider a body subjected to several forces F_1, F_2, F_3 etc as shown in Fig. 3.122. This force system is to be replaced by a single force acting through the point O along with a couple.

Let R be the resultant force acting through O and M_O be the resultant moment which is equal to the sum of the moments of the forces F_1, F_2 etc about O. Then,

$R = F_1 + F_2 + F_3.$

$R_x = \Sigma F_x$, $R_y = \Sigma F_y$ and $R_z = \Sigma F_Z$

Therefore, $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$

Sum of moments of all the forces about O should be equal to the moment of resultant about O.

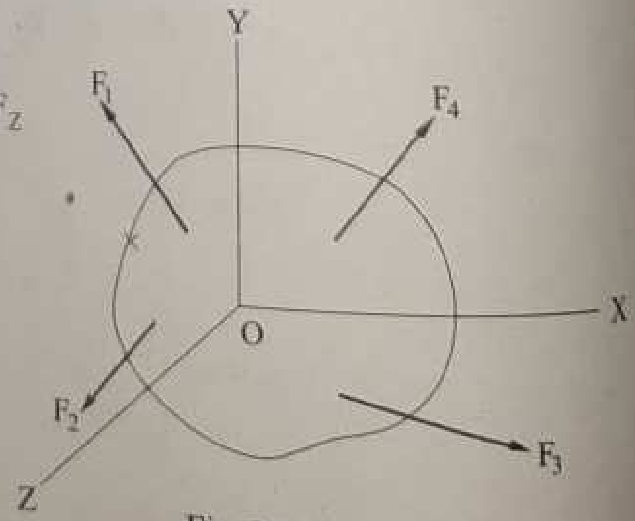

Fig. 3.122

$$\text{ie., } \Sigma M_o = (M_R)_o$$

$$\text{or } \Sigma M_x = (M_R)_x$$

$$\Sigma M_y = (M_R)_y \quad \text{and}$$

$$\Sigma M_z = (M_R)_z$$

$$(M_R)_O = \sqrt{\Sigma M_x^2 + \Sigma M_y^2 + \Sigma M_z^2}$$

Example 3.64

A rectangular concrete slab supports loads at its four corners as shown in Fig. 3.123. Determine the resultant of these forces and the point of application of the resultant.

Solution.

$$\vec{F}_O = -125\,\vec{j},\quad \vec{F}_A = -250\,\vec{j},\quad \vec{F}_B = -150\,\vec{j} \text{ and } \vec{F}_C = -100\,\vec{j}$$

$$\text{Resultant } \vec{R} = \Sigma\vec{F} = \vec{F}_O + \vec{F}_A + \vec{F}_B + \vec{F}_C$$

$$= -125\,\vec{j} + (-250\,\vec{j}) + (-150\,\vec{j}) + (-100\,\vec{j})$$

$$= -625\,\vec{j}.$$

Position vector of point O, $\vec{r}_o = 0$

Position vector of point A, $\vec{r}_A = 4\,\vec{i}$

Position vector of point B, $\vec{r}_B = 4\,\vec{i} + 5\,\vec{k}$

Position vector of point C, $\vec{r}_C = 5\,\vec{k}$

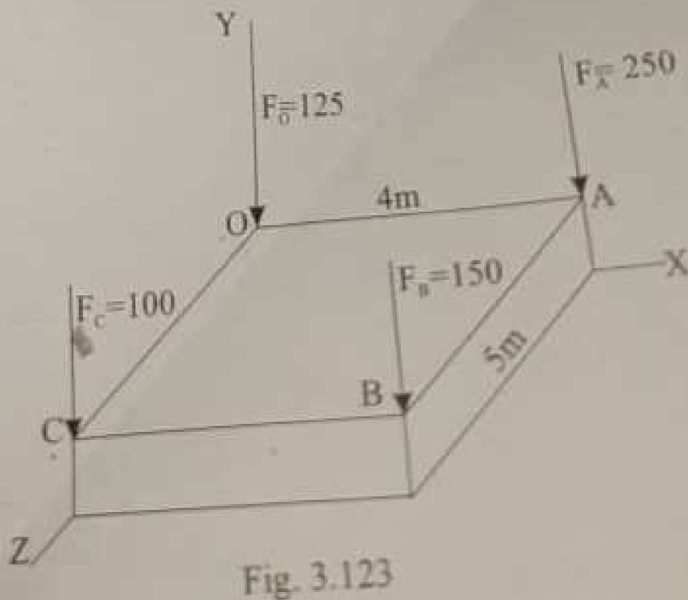

Fig. 3.123

Moment of $\vec{F}_D$ about O is zero.

Moment of $\vec{F}_A$ about O is $\vec{r}_A \times \vec{F}_A$

$$= 4\,\vec{i} \times (-250\,\vec{j})$$
$$= -1000\,(\vec{i} \times \vec{j})$$
$$= -1000\,\vec{k}.$$

Moment of $\vec{F}_B$ about O is $\vec{r}_B \times \vec{F}_B$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 0 & 5 \\ 0 & -150 & 0 \end{vmatrix} = 750\,\vec{i} - 600\,\vec{k}$$

Moment of $\vec{F}_C$ about O is $\vec{r}_C \times \vec{F}_C$

$$= 5\,\vec{k} \times (-100\,\vec{j}) = -500\,(\vec{k} \times \vec{j})$$
$$= 500\,\vec{i}.$$

Sum of moment of all the forces about O,

$$\Sigma M_O = 0 + (-1000\,\vec{k}) + (750\,\vec{i} - 600\,\vec{k}) + 500\,\vec{i}$$
$$= 1250\,\vec{i} - 1600\,\vec{k}.$$

Let co-ordinates of point D, the point of application of resultant, be x_D and z_D.

Position vector of D is, $\vec{r}_D = x_D\,\vec{i} + z_D \times \vec{k}$.

$$\vec{F}_D = \vec{R} = -625\,\vec{j}$$

Moment of resultant about O is $\vec{r}_D \times \vec{R}$

$$= (x_D\,\vec{i} + z_D\,\vec{k}) \times (-625\,\vec{j})$$
$$= -625\,x_D\,(\vec{i} \times \vec{j}) - 625\,z_D\,(\vec{k} \times \vec{j})$$
$$= -625\,x_D\,\vec{k} - 625\,z_D\,(-\vec{i})$$
$$= -625\,x_D\,\vec{k} + 625\,z_D\,\vec{i}.$$

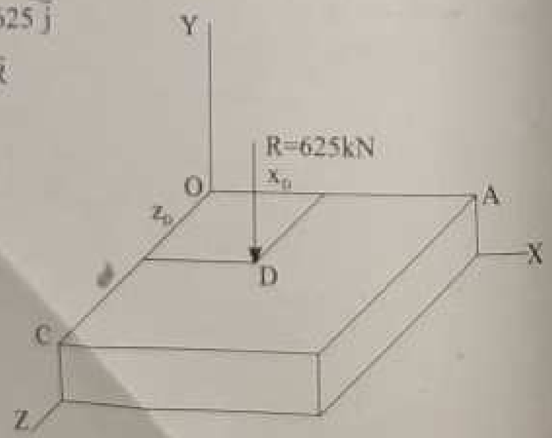

Fig. 3.124

Equating this moment of resultant about O and the sum of moment of all the forces about O,

$$\vec{r}_D \times \vec{R} = \Sigma M_O.$$

$$625\, z_D\, \vec{i} - 625\, x_D\, \vec{k} = 1250\, \vec{i} - 1600\, \vec{k}$$

$$625\, z_D = 1250$$

$$z_D = 2\ \text{m}$$

$$-625\, x_D = -1600$$

$$x_D = \frac{1600}{625} = 2.56\ \text{m}$$

Example 3.65

A rectangular block is subjected to three forces as shown in Fig 3.125. Reduce them to an equivalent force couple system acting at A.

Solution.

Co-ordinates of various points are

A (0, 0,0); P (0,1,0) ; Q (3,1,0) ; R (3,1,2) ; S (0,1,2) ; D (0,0,2)

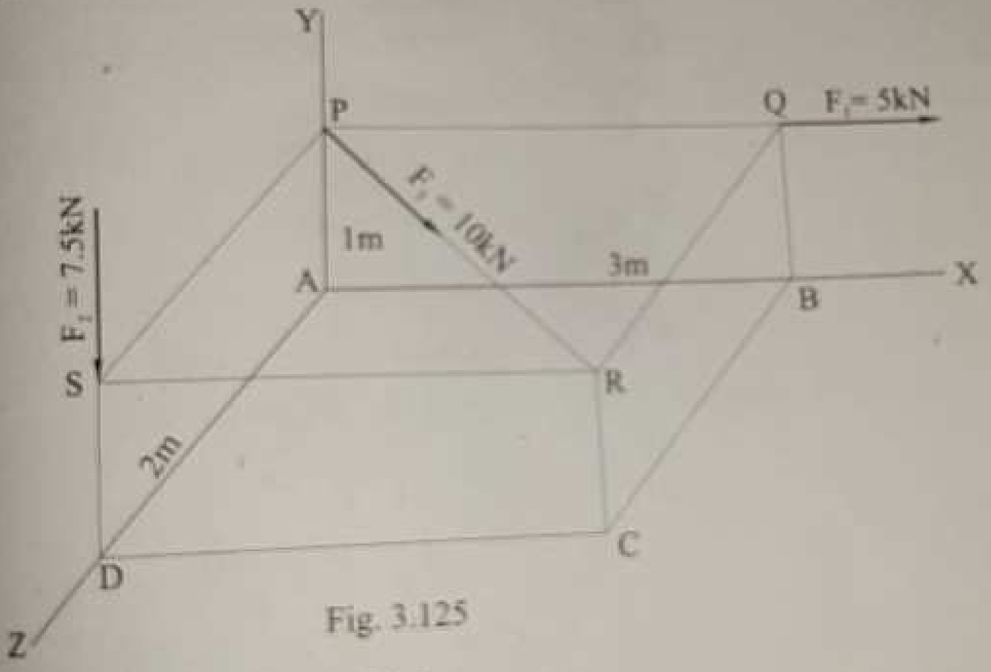

Fig. 3.125

Unit vector in the direction of PQ is,

$$= \frac{(3-0)i + (1-1)j - 0k}{\sqrt{3^2 + (1-1)^2 + 0^2}} = \frac{3i}{3} =$$

$$F_1 = 5\,(i + 0\,j + 0\,k)$$

Unit vector in the direction of PR,

$$= \frac{(3-0)i + (1-1)j + (2-0)k}{\sqrt{3^2 + 0^2 + 2^2}}$$

$$= \frac{3i + 0j + 2k}{\sqrt{13}} = 0.832\,i + 0\,j + 0.554\,k.$$

$$F_3 = 10\,(0.832\,i + 0\,j + 0.554\,k)$$

$$= 8.32\,i + 0\,j + 5.54\,k.$$

Unit vector in the direction of SD,

$$= \frac{(0-0)i + (0-1)j + (2-2)k}{\sqrt{0^2 + 1^2 - 0^2}} = 0i - j + 0k$$

$$F_2 = 7.5\,(0\,i - j + 0\,k)$$

$$= 0\,i - 7.5\,j + 0\,k.$$

$$\text{Resultant } R = F_1 + F_2 + F_3$$

$$= (5 + 0 + 8.32)\,i + (0 - 7.5 + 0)\,j + (0 + 5.54 + 0)\,k$$

$$R = 13.32\,i - 7.5\,j + 5.54\,k$$

The resultant moment about A is obtained by summing the moments of F_1, F_2 and F_3 about A

$$M_A = r_{AP} \times F_1 + r_{AP} \times F_3 + r_{AD} \times F_2$$

$$r_{AP} = (0 - 0)\,i + (1 - 0)\,j + (0 - 0)\,z$$

$$= 0\,i + j + 0\,k.$$

$$r_{AD} = (0 - 0)\,i + (0 - 0)\,j + (2 - 0)\,k$$

$$= 0\,i + 0\,i + 2\,k.$$

$$r_{AP} \times F_1 = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 5 & 5 & 0 \end{vmatrix} = 0i + 0j - 5k$$

i) –
ii) Expan...

$$r_{AP} \times F_3 = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 8.32 & 0 & 5.54 \end{vmatrix} = 5.54i + 0j - 8.32k$$

$$r_{AD} \times F_2 = \begin{vmatrix} i & j & k \\ 0 & 0 & 2 \\ 0 & -7.5 & 0 \end{vmatrix} = 15i + 0j - 0k$$

$$M_A = r_{AP} \times F_1 + r_{AP} \times F_3 + r_{AD} \times F_2$$

$$= (0i + 0j - 5k) + (5.54i + 0j - 8.32k) + (15i + 0j - 0k)$$

$$= (20.54\,i + 0\,j + 13.32\,k)\ \text{kNm.}$$

3.14. Equilibrium equation of non-concurrent forces in space.

A body in space is said to be in equilibrium when the resultant force and the resultant moment due to the forces acting on the body are zero.

Resultant force $R = R_x i + R_y j + R_z k = 0$

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0$$

Resultant moment,

$$Mo = M_x i + M_y j + M_z k = 0$$

$$\Sigma M_x i + \Sigma M_y j + \Sigma M_z k = 0$$

$$\Sigma M_x = 0$$

$$\Sigma M_y = 0$$

$$\Sigma M_z = 0$$

Example 3.66

Find the reaction and moment at support O for the structure shown in Fig. 3.126

Solution.

$$\vec{F}_A = 10\,\vec{j}, \quad F_B = 20\,\vec{i}, \quad F_C = 30\,\vec{k}$$

Position vector $\vec{r}_A = 3\vec{i}$

Position vector $\vec{r}_B = 3\vec{i} + 4\vec{k}$

Position vector $\vec{r}_C = 3\vec{i} - 2\vec{j} + 4\vec{k}$

Let $\vec{R}$ be the reaction at O.

Its components are R_x, R_y, R_z.

For the sum of forces acting on the structure to be zero, $R + F_A + F_B + F_C = 0$

$$R_x\,\vec{i} + R_y\,\vec{j} + R_z\,\vec{k} + F_A + F_B + F_C = 0$$

$$R_x\,\vec{i} + R_y\,\vec{j} + R_z\,\vec{k} - 10\,\vec{j} + 20\,\vec{i} + 30\,\vec{k} = 0$$

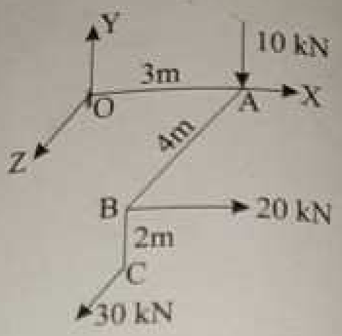

Fig. 3.126

$$(R_x + 20)\vec{i} = 0$$

$$R_x = -20\text{ kN}$$

$$R_y\,\vec{j} - 10\,\vec{j} = 0$$

$$R_y = 10\text{ kN}$$

$$(R_z + 30)\,\vec{k} = 0$$

$$R_z + 30 = 0$$

$$R_z = -30\text{ kN}$$

The resultant reaction at O, $\sqrt{R_x^{\,2} + R_y^{\,2} + R_z^{\,2}}$

$$R = \sqrt{20^2 + 10^2 + 30^2}$$

$$= 37.42\text{ kN}$$

Inclination of the resultant with X, Y and Z axes are

$$\theta_x = \cos^{-1}\left(\frac{R_x}{R}\right) = \cos^{-1}\frac{20}{37.42} = 57.69^0$$

$$\theta_y = \cos^{-1}\frac{R_y}{R}) = \cos^{-1}\frac{10}{37.42} = 74.5^0$$

$$\theta_z = \cos^{-1}\frac{R_z}{R} = \cos^{-1}\frac{30}{37.42} = 36.71^0$$

Moment of force $\bar{F}_A$ about O is $\bar{r}_A \times \bar{F}_A$

$$= 3\,\bar{i} \times -10\,\bar{j}$$

$$= -30\,(\bar{i} \times \bar{j})$$

$$= -\,30\bar{k}$$

Moment of $\bar{F}_B$ about O is

$$\bar{r}_B \times \bar{F}_B = (3\,\bar{i} + 4\,\bar{k}) \times 20\,\bar{i}$$

$$= 60\,(\vec{i} \times \vec{i}) + 80\,(\vec{k} \times \vec{i})$$

$$= 80\,\vec{j}.$$

Moment of $\vec{F}_C$ about O is $\vec{r}_C \times \vec{F}_C$

$$= (3\,\vec{i} - 2\,\vec{j} + 4\,\vec{k}) \times 30\,\vec{k}$$

$$= 90\,(\vec{i} \times \vec{k}) - 60\,(\vec{j} \times \vec{k}) + 120\,(\vec{k} \times \vec{k})$$

$$= -90\,\vec{j} - 60\,\vec{i} + 0$$

$$= -60\,\vec{i} - 90\,\vec{j}.$$

Sum of moments of all the forces at O,

$$= -30\,\vec{k} + 80\,\vec{j} - 60\,\vec{i} - 90\,\vec{j}$$

$$= -60\,\vec{i} - 10\,\vec{j} - 30\,\vec{k}.$$

Let $\vec{M}$ be the moment at O,

$$\vec{M} = M_x\,\vec{i} + M_y\,\vec{j} + M_z\,\vec{k}.$$

Since for equilibrium $\Sigma M = 0$

$$M_x\,\vec{i} + M_y\,\vec{j} + M_z\,\vec{k} + [-60\,\vec{i} - 10\,\vec{j} - 30\,\vec{k}] = 0$$

$$(M_x - 60)\,\vec{i} + (M_y - 10)\,\vec{j} + (M_z - 30)\,\vec{k} = 0.$$

$M_x = 60$ k N m, $\quad M_y = 10$ k N m, $\quad Mz = 30$ k N m

$$\text{Moment at O, } M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

$$= \sqrt{60^2 + 10^2 + 30^2} = 67.82\text{kNm. c.c.w.}$$

3.16. Mass moment of inertia.

Mass is the quantitative measure of the resistance to change the motion of a body. The inertia of a body is the property by virtue of which it resists any change in its state of rest or of uniform motion. Translatory inertia is defined as mass and rotational inertia is known as moment of inertia.

Consider two bodies having the same mass but of different shape as shown in Fig. 3.127

Resistance to rotation about the axis is different for the two bodies, eventhough they have the same mass. The resistance to rotation about the axis AB depends on the expression $\int dm\,x^2$, where dm is an elemental mass of the body at a distance x from the AB axis. Thus the moment of inertia of a body with respect to a given axis is defined by the integral,

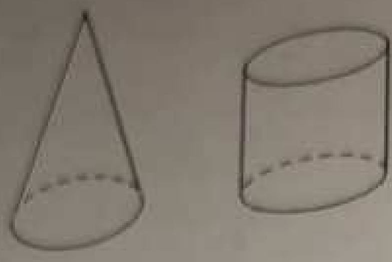

Fig. 3.127

$I = \int dm\, x^2$. The unit of mass moment of inertia is kg-m^2. The radius of gyration of the body is expressed as $k = \sqrt{\frac{I}{m}}$, where m is the mass of body. The moment of inertia of a body about an axis at a distance d and parallel to centroidal axis is equal to the sum of moment of inertia about centroidal axis and product of mass and square of distance between the parallel axes.

$$I = I_G + md^2$$

Mass moment of inertia of a ring of radius R

Let A be the cross sectional area of the ring and ρ be the mass density of ring material. Consider an elemental length of ring, dl. Volume of ring of this elemental length is $A \times dl$. Mass of this elemental volume is $\rho \times A \times dl$. Second moment of this mass about the ZZ axis is $\rho \times A \times dl \times R^2$. Moment of inertia of the ring about ZZ axis,

$$I_{zz} = \int_0^{2\pi R} \rho A\, dl \times R^2 = R^2 \rho A \left[l\right]_0^{2\pi R}$$

$$= R^2 \rho A \times 2\pi R$$

$$= R^2\, 2\pi R A \rho$$

$$I_{zz} = mR^2$$

$I_{zz} = I_{xx} + I_{yy}$ Because of symmetry $I_{xx} = I_{yy}$

$$I_{zz} = I_{xx} + I_{yy} = 2\, I_{xx}$$

$$I_{xx} = I_{yy} = \frac{MR^2}{2}$$

Mass moment of inertia of disc.

Consider an elemental ring of radial thickness dr at a distance r from the centre.

Volume of the ring $= 2\pi r\, dr \times t$

Mass of elemental ring $= 2\pi r\, dr\, t \times \rho$

Moment of inertia of the plate about the axis through the centre and perpendicular to the plane of plate is $\int dm\, r^2$

$$I_{zz} = \int_0^R 2\pi r\, dr\, t\, \rho r^2$$

$$= 2\pi t\rho \int_0^R r^3\, dr = 2\pi t\rho \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{2\pi t\rho}{4} R^4 = (\pi R^2 t\rho)\frac{R^2}{2}$$

$$= \frac{mR^2}{2}$$

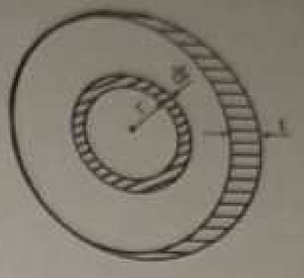

Fig. 3.128

Polar moment of inertia $I_{ZZ} = I_{XX} + I_{YY} = \frac{mR2}{2}$

$$I_{XX} = I_{yy} = \frac{I_{ZZ}}{2} = \frac{mR2}{4}$$

Mass moment of inertia of cylinder

Consider an elemental circular disc of thickness dy at a distance y from the centroidal XX axis of the cylinder.

Mass of the element, $dm = \pi R^2\, dy\, \rho$.

Moment of inertia thin circular disc about its centroidal XX axis, $dI = \frac{dm\, R^2}{4}$

$$dI_{XX} = dI + (dm)\, y^2$$

$$dI_{XX} = \left[\frac{dmR^2}{4} + dm\, y^2\right]$$

$$= \pi R^2 dy \frac{\rho R^2}{4} + \pi R^2 dy \times \rho y^2$$

$$I_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{\pi R^2}{4} \rho\right) dy + \pi R^2 \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy$$

$$= 2\pi \frac{R^4}{4} \rho [y]_0^{\frac{h}{2}} + 2.\pi R^2 \rho \left[\frac{y^3}{3}\right]_0^{\frac{h}{2}}$$

$$= \frac{\pi R^4}{2} \frac{\rho h}{2} + \frac{2\pi R^2}{3} \rho \left[\frac{h}{2}\right]^3$$

$$= \frac{\pi R^2 h \rho}{4} \left(R^2 + \frac{h^2}{3}\right)$$

$$= \frac{M}{4} \left(\frac{3R^2 + h^2}{3}\right)$$

$$= \frac{M}{12} (3R^2 + h^2)$$

$$I_{zz} = I_{xx} = \frac{M}{12} (3R^2 + h^2)$$

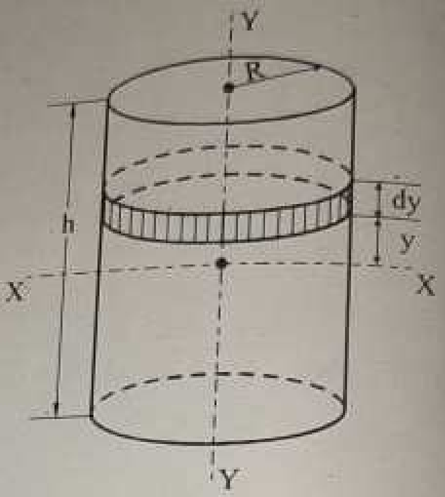

Fig. 3.142